22.51 Problem Set 1 (due Mon, Sept. 10)

1 Least-Action Principle (25 pt)

Question: Action has the unit of energy·time, therefore a plausible expression for the action of a photon would be,

$$A = \int_{\mathbf{x_1}(t_1)}^{\mathbf{x_2}(t_2)} \hbar\omega dt, \tag{1}$$

because $\hbar\omega$, the energy of a photon, is an apparent energy scale. It is well-known that the speed of light in a material is $v = c/n$ where $n \geq 1$ is the refraction index. Therefore,

$$dt = \frac{ds}{v} = \frac{nds}{c} \quad \longrightarrow \quad A = \frac{\hbar\omega}{c}\int_{\mathbf{x_1}(t_1)}^{\mathbf{x_2}(t_2)} nds, \tag{2}$$

where ds is the differential distance, and we have used the knowledge that ω remains constant.

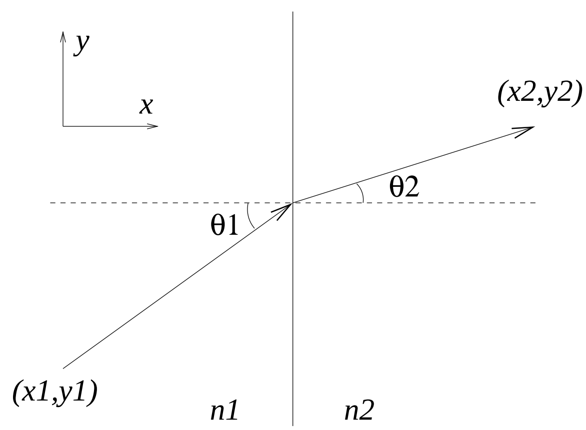

Figure 1: Least-Action Principle.

Now suppose we have an interface between two materials of constant refraction indices n_1,n_2, prove Snell's law,

$$n_1 \sin(\theta_1) \quad = \quad n_2 \sin(\theta_2), \tag{3}$$

that is, such a path would minimize A among all feasible paths $\{\mathbf{x}(t)\}$ that start at $\mathbf{x}(t_1) = (x_1, y_1)$ and end at $\mathbf{x}(t_2) = (x_2, y_2)$.

Answer: Let $x = x_i$ be the interface. All feasible light paths will intersect the interface at some point, say (x_i, y_i). Among light paths that crosses a certain (x_i, y_i), straight lines linking (x_1, y_1) to (x_i, y_i), and then (x_i, y_i) to (x_2, y_2) are the shortest, the total action for which would be (we omit the $\hbar\omega/c$ constant factor for the sake of simplicity),

$$A = n_1\sqrt{(x_i - x_1)^2 + (y_i - y_1)^2} + n_2\sqrt{(x_i - x_2)^2 + (y_i - y_2)^2}. \tag{4}$$

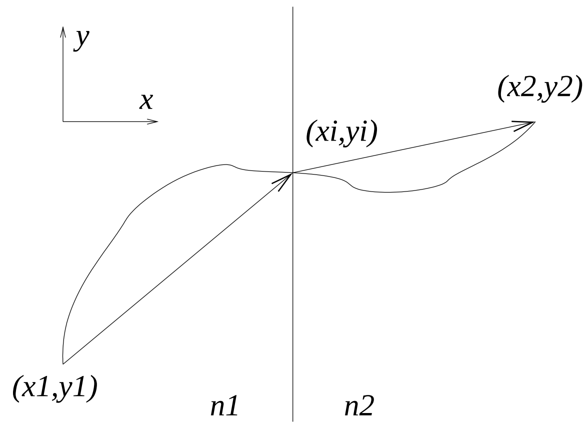

Figure 2: Least-Action Principle.

We then minimize with respect to y_i, the only degree of freedom left,

$$\frac{n_1(y_i - y_1)}{\sqrt{(x_i - x_1)^2 + (y_i - y_1)^2}} + \frac{n_2(y_i - y_2)}{\sqrt{(x_i - x_2)^2 + (y_i - y_2)^2}} = 0. \tag{5}$$

This is just the Snell's law.

Bonus Question (10 pt): Suppose we have a graded material with $n(x, y) = 1 + x$. Let $\mathbf{x}(t_1) = (0, 0)$, $\mathbf{x}(t_2) = (0.5, 0.3)$, calculate the light path $y(x)$.

Answer: The action as a functional of $y(x)$ is,

$$A[y] = \int_{x_1}^{x_2} n ds = \int_{x_1}^{x_2} n(x)\sqrt{dx^2 + dy^2} = \int_{x_1}^{x_2} (1 + x)\sqrt{1 + y'^2} dx, \tag{6}$$

$$\begin{aligned} 0 = \delta A \quad &= \int_{x_1}^{x_2} (1 + x)\frac{y'\delta y'}{\sqrt{1 + y'^2}} dx, \\ &= \int_{x_1}^{x_2} \left[\frac{d}{dx}\left((1 + x)\frac{y'\delta y}{\sqrt{1 + y'^2}}\right) - \frac{d}{dx}\left(\frac{(1 + x)y'}{\sqrt{1 + y'^2}}\right)\delta y\right] dx. \end{aligned} \tag{7}$$

The first term always vanishes since,

$$\delta y|_{x_1} = 0, \quad \delta y|_{x_2} = 0. \tag{8}$$

The second term gives zero for all feasible δy if and only if,

$$\frac{(1+x)y'}{\sqrt{1+y'^2}} \;=\; C, \tag{9}$$

for all x, where C is a constant. This means,

$$\frac{y'}{\sqrt{1+y'^2}} \;=\; \frac{C}{1+x}, \tag{10}$$

$$\frac{1}{1+y'^2} \;=\; 1-\frac{C^2}{(1+x)^2}, \tag{11}$$

$$y' \;=\; \sqrt{\frac{1}{1-\frac{C^2}{(1+x)^2}}-1} \;=\; \frac{C}{\sqrt{(1+x)^2-C^2}}, \tag{12}$$

Using Maple, one obtains,

$$0.3 = y(0.5)-y(0) = \int_0^{0.5}\frac{C}{\sqrt{(1+x)^2-C^2}} \quad\longrightarrow\quad C = .6296364651. \tag{13}$$

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> int(C/sqrt((1+x)^2-C^2),x=0..0.5);
```

$$C\ln(1.5+\sqrt{2.25-1.\,C^2})-1.\,\ln(1.+\sqrt{-1.\,(C-1.)\,(C+1.)})\,C$$

```
> fsolve(C*ln(1.5+sqrt(2.25-1.*C^2))-1.*ln(1.+sqrt(-1.*(C-1.)*(C+1.)
  ))*C - 0.3, C);
```

$$.6296364651$$

```
> C := .6296364651;
```

$$C := .6296364651$$

```
> with(plots):
> p:= dsolve({D(y)(x) = C/sqrt((1+x)^2-C^2), y(0)=0}, type=numeric,
  range=0..0.5):
> odeplot(p,[x,y(x)],0..0.5,refine=3);
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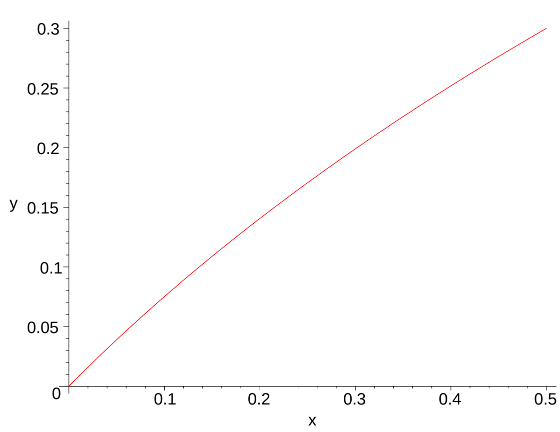

Figure 3: Least-Action Principle.

2 Double Pendulum (25 pt)

Question: Lagrangian mechanics is formally equivalent to Newtonian mechanics (the free body diagram and torque analysis approach), but tends to be less susceptible to human mistakes, especially when the system is complex or has constraints. Consider a double pendulum shown in Fig. 4, composed of two uniform rods of lengths l_1, l_2 and total masses m_1, m_2 (mass is uniformly distributed on each rod),

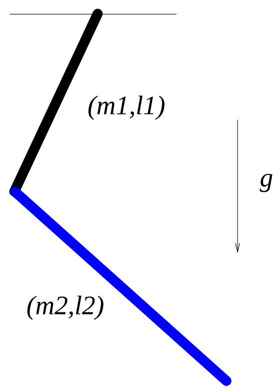

Figure 4: Double Pendulum.

(**a**). Choose variables $\{q_k\}$ which describes the system's degrees of freedom "most naturally".

(**b**). The gravitational constant is g. Express the potential energy V in $\{q_k\}$.

(**c**). Express the kinetic energy T in $\{q_k, \dot{q}_k\}$.

(**d**). Write down $\mathcal{L} = T - V$. Derive the equation of motion.

(**e**). What are the small oscillation normal-mode frequencies? Describe those modes in physical terms.

(**f**). Determine the conjugate momenta $\{p_k\}$ from $\mathcal{L}$.

(**g**). Write down the Hamiltonian $\mathcal{H} = \sum_k \dot{q}_k p_k - \mathcal{L}$. Reexpress $\mathcal{H}$ in $\{q_k, p_k\}$ instead of $\{q_k, \dot{q}_k\}$.

(**h**). Rederive the equations of motion using $\dot{q}_k = \partial\mathcal{H}/\partial p_k$, $\dot{p}_k = -\partial\mathcal{H}/\partial q_k$, and check whether they are equivalent to (**d**).

(**i**). Rationalize (rederive) the equations of motion using Newtonian mechanics.

Answer:

(**a**). θ_1, θ_2.

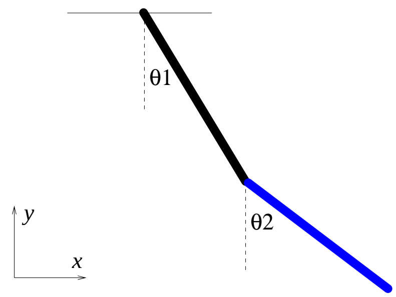

Figure 5: Double Pendulum.

(**b**). The center of mass of the first rod is at,

$$(x_1, y_1) \;=\; \left(\frac{l_1}{2}\sin\theta_1,\;\; -\frac{l_1}{2}\cos\theta_1\right). \tag{14}$$

The center of mass of the second rod is at,

$$(x_2, y_2) \;=\; \left(l_1\sin\theta_1 + \frac{l_2}{2}\sin\theta_2,\;\; -l_1\cos\theta_1 - \frac{l_2}{2}\cos\theta_2\right). \tag{15}$$

Therefore the total potential energy is,

$$V \;=\; m_1 g y_1 + m_2 g y_2 \;=\; -\frac{(m_1 + 2m_2) g l_1 \cos\theta_1}{2} - \frac{m_2 g l_2 \cos\theta_2}{2}. \tag{16}$$

(**c**). From (**b**), we have,

$$(\dot{x}_1, \dot{y}_1) \;=\; \left(\frac{l_1}{2}\cos\theta_1\dot{\theta}_1,\;\; \frac{l_1}{2}\sin\theta_1\dot{\theta}_1\right), \tag{17}$$

$$(\dot{x}_2, \dot{y}_2) \;=\; \left(l_1\cos\theta_1\dot{\theta}_1 + \frac{l_2}{2}\cos\theta_2\dot{\theta}_2,\;\; l_1\sin\theta_1\dot{\theta}_1 + \frac{l_2}{2}\sin\theta_2\dot{\theta}_2\right). \tag{18}$$

Therefore the center of mass kinetic energy is,

$$\begin{aligned} T_1 = \;& \frac{m_1}{2}(\dot{x}_1^2 + \dot{y}_1^2) + \frac{m_2}{2}(\dot{x}_2^2 + \dot{y}_2^2) \\ = \;& \frac{m_1}{2}\left(\frac{l_1^2}{4}\dot{\theta}_1^2\right) + \frac{m_2}{2}\left(l_1^2\dot{\theta}_1^2 + \frac{l_2^2}{4}\dot{\theta}_2^2 + l_1 l_2\cos(\theta_1 - \theta_2)\dot{\theta}_1\dot{\theta}_2\right). \end{aligned} \tag{19}$$

In addition to the center of mass translational kinetic energies, there are also rotational kinetic energies. The sum is,

$$T_2 \quad = \quad \frac{I_1}{2}\dot{\theta}_1^2 + \frac{I_2}{2}\dot{\theta}_2^2 \quad = \quad \frac{m_1 l_1^2 \dot{\theta}_1^2}{24} + \frac{m_2 l_2^2 \dot{\theta}_2^2}{24}. \tag{20}$$

Therefore the total kinetic energy is,

$$T \;=\; \frac{(m_1 + 3m_2) l_1^2 \dot{\theta}_1^2}{6} + \frac{m_2 l_2^2 \dot{\theta}_2^2}{6} + \frac{m_2 l_1 l_2 \cos(\theta_1 - \theta_2)\dot{\theta}_1 \dot{\theta}_2}{2}, \tag{21}$$

(**d**).

$$\begin{aligned} \mathcal{L}(\theta_1, \theta_2, \dot{\theta}_1, \dot{\theta}_2) \;=\; & \frac{(m_1 + 3m_2) l_1^2 \dot{\theta}_1^2}{6} + \frac{m_2 l_2^2 \dot{\theta}_2^2}{6} + \frac{m_2 l_1 l_2 \cos(\theta_1 - \theta_2)\dot{\theta}_1 \dot{\theta}_2}{2} \\ & + \frac{(m_1 + 2m_2) g l_1 \cos\theta_1}{2} + \frac{m_2 g l_2 \cos\theta_2}{2}. \end{aligned} \tag{22}$$

$$\frac{\partial \mathcal{L}}{\partial \dot{\theta}_1} = \frac{(m_1 + 3m_2) l_1^2 \dot{\theta}_1}{3} + \frac{m_2 l_1 l_2 \cos(\theta_1 - \theta_2)\dot{\theta}_2}{2}, \tag{23}$$

$$\frac{\partial \mathcal{L}}{\partial \theta_1} \;=\; -\frac{m_2 l_1 l_2 \sin(\theta_1 - \theta_2)\dot{\theta}_1 \dot{\theta}_2}{2} - \frac{(m_1 + 2m_2) g l_1 \sin\theta_1}{2} \tag{24}$$

$$\Longrightarrow \qquad \frac{d}{dt}\left(\frac{\partial \mathcal{L}}{\partial \dot{\theta}_1}\right) \;=\; \frac{\partial \mathcal{L}}{\partial \theta_1}. \tag{25}$$

$$\frac{\partial \mathcal{L}}{\partial \dot{\theta}_2} = \frac{m_2 l_2^2 \dot{\theta}_2}{3} + \frac{m_2 l_1 l_2 \cos(\theta_1 - \theta_2)\dot{\theta}_1}{2}, \tag{26}$$

$$\frac{\partial \mathcal{L}}{\partial \theta_2} \;=\; \frac{m_2 l_1 l_2 \sin(\theta_1 - \theta_2)\dot{\theta}_1 \dot{\theta}_2}{2} - \frac{m_2 g l_2 \sin\theta_2}{2}. \tag{27}$$

$$\Longrightarrow \qquad \frac{d}{dt}\left(\frac{\partial \mathcal{L}}{\partial \dot{\theta}_2}\right) \;=\; \frac{\partial \mathcal{L}}{\partial \theta_2}. \tag{28}$$

(**e**). When $\theta_1, \theta_2 \sim 0$, keeping only the linear order terms,

$$\frac{(m_1 + 3m_2) l_1^2 \ddot{\theta}_1}{3} + \frac{m_2 l_1 l_2 \ddot{\theta}_2}{2} \;=\; -\frac{(m_1 + 2m_2) g l_1 \theta_1}{2}, \tag{29}$$

$$\frac{m_2 l_2^2 \ddot{\theta}_2}{3} + \frac{m_2 l_1 l_2 \ddot{\theta}_1}{2} \;=\; -\frac{m_2 g l_2 \theta_2}{2}, \tag{30}$$

Or,

$$(2m_1 + 6m_2)l_1^2\ddot{\theta}_1 + 3m_2 l_1 l_2 \ddot{\theta}_2 = -(3m_1 + 6m_2)gl_1\theta_1, \tag{31}$$

$$2m_2 l_2^2\ddot{\theta}_2 + 3m_2 l_1 l_2\ddot{\theta}_1 = -3m_2 g l_2\theta_2, \tag{32}$$

To determine the normal modes, let us multiply (32) by α and add it onto (31),

$$[(2m_1+6m_2)l_1^2+3m_2l_1l_2\alpha]\ddot{\theta}_1+[3m_2l_1l_2+2m_2l_2^2\alpha]\ddot{\theta}_2 = -(3m_1+6m_2)gl_1\theta_1-3m_2gl_2\alpha\theta_2. \tag{33}$$

For properly chosen α, the coefficient ratio of $\ddot{\theta}_1$, $\ddot{\theta}_2$ on LHS would equal to the coefficient ratio of θ_1, θ_2 on RHS, allowing one to consolidate a single equation out of two coupled equations,

$$\frac{(2m_1+6m_2)l_1^2+3m_2l_1l_2\alpha}{3m_2l_1l_2+2m_2l_2^2\alpha} = \frac{(3m_1+6m_2)gl_1}{3m_2gl_2\alpha}, \tag{34}$$

for which there are two roots, $\alpha_+ > 0$ and $\alpha_- < 0$. And the vibrational frequencies of the two normal modes would then be,

$$\omega_- = \sqrt{\frac{(3m_1+6m_2)gl_1}{(2m_1+6m_2)l_1^2+3m_2l_1l_2\alpha_+}}, \quad \omega_+ = \sqrt{\frac{(3m_1+6m_2)gl_1}{(2m_1+6m_2)l_1^2+3m_2l_1l_2\alpha_-}}. \tag{35}$$

The low frequency (ω_-) mode $(3m_1+6m_2)gl_1\theta_1 + 3m_2gl_2\alpha_+\theta_2$ corresponds to the case of θ_1,θ_2 swinging in phase. The high frequency (ω_+) mode $(3m_1+6m_2)gl_1\theta_1 + 3m_2gl_2\alpha_-\theta_2$ corresponds to the case of θ_1,θ_2 swinging anti-phase.

(**f**).

$$p_1 \equiv \frac{\partial\mathcal{L}}{\partial\dot{\theta}_1} = \frac{(m_1+3m_2)l_1^2\dot{\theta}_1}{3} + \frac{m_2l_1l_2\cos(\theta_1-\theta_2)\dot{\theta}_2}{2}, \tag{36}$$

$$p_2 \equiv \frac{\partial\mathcal{L}}{\partial\dot{\theta}_2} = \frac{m_2l_2^2\dot{\theta}_2}{3} + \frac{m_2l_1l_2\cos(\theta_1-\theta_2)\dot{\theta}_1}{2}, \tag{37}$$

from which we can solve for $\dot{\theta}_1$, $\dot{\theta}_2$ (using Maple) as,

$$\dot{\theta}_1 = \frac{-12l_2p_1 + 18l_1\cos(\theta_1-\theta_2)p_2}{l_2l_1^2(-4m_1-12m_2+9m_2\cos^2(\theta_1-\theta_2))} \tag{38}$$

$$\dot{\theta}_2 = \frac{-12l_1m_1p_2 - 36l_1m_2p_2 + 18p_1m_2l_2\cos(\theta_1-\theta_2)}{m_2l_2^2l_1(-4m_1-12m_2+9m_2\cos^2(\theta_1-\theta_2))} \tag{39}$$

(**g**).

$$\mathcal{H} \equiv \dot{\theta}_1p_1 + \dot{\theta}_2p_2 - \mathcal{L}$$

$$
\begin{aligned}
= \quad & \frac{(m_1+3m_2)l_1^2\dot\theta_1^2}{6} + \frac{m_2 l_2^2\dot\theta_2^2}{6} + \frac{m_2 l_1 l_2\cos(\theta_1-\theta_2)\dot\theta_1\dot\theta_2}{2} \\
& -\frac{(m_1+2m_2)gl_1\cos\theta_1}{2} - \frac{m_2 g l_2\cos\theta_2}{2}. \qquad (40)
\end{aligned}
$$

Plugging (38),(39) into above, and using Maple,

$$
\begin{aligned}
\mathcal{H} \quad = \quad & \frac{-6m_2 l_2^2 p_1^2 - 6(m_1+3m_2)l_1^2 p_2^2 + 18 m_2 l_1 l_2 \cos(\theta_1-\theta_2)p_1 p_2}{m_2 l_2^2 l_1^2(-4m_1 - 12m_2 + 9m_2\cos^2(\theta_1-\theta_2))} \\
& -\frac{(m_1+2m_2)gl_1\cos\theta_1}{2} - \frac{m_2 g l_2\cos\theta_2}{2}. \qquad (41)
\end{aligned}
$$

(**h**). It is straightforward to verify that $\dot\theta_1 = \partial\mathcal{H}/\partial p_1$, $\dot\theta_2 = \partial\mathcal{H}/\partial p_2$ indeed agrees with (38), (39). Similarly, one can verify that $\dot p_1 = -\partial\mathcal{H}/\partial\theta_1$, $\dot p_2 = -\partial\mathcal{H}/\partial\theta_2$ indeed agrees with (25), (28).

(**i**). That is pretty tricky!

3 Collision Cross-Section of Hard Spheres (25 pt)

Question: This course is mainly concerned with probing materials by radiation. Assume both the probe and the target are hard spheres of mass m and radius a, and assume they interact by frictionless elastic collision only.

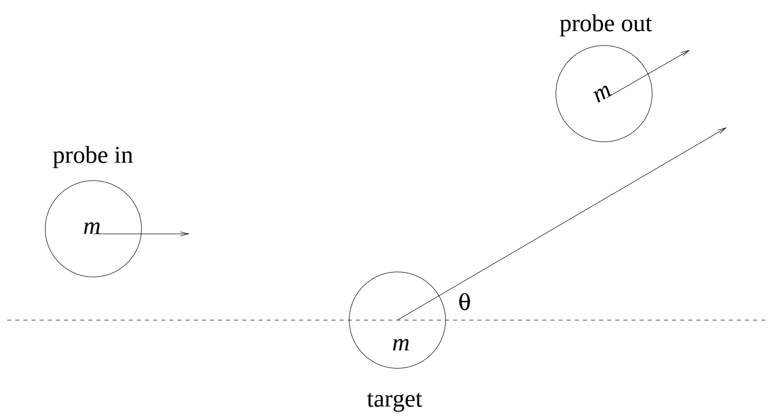

Figure 6: Collision cross-section.

A central concept of the course is the double differential cross-section $d^2\sigma/d\Omega dE$, which has the following interpretation: imagine immersing a single target sphere in a uniform flux f of probing spheres, where f is the number of incoming probes per unit cross-sectional area per unit time, and suppose the incoming probes all have the same kinetic energy E_0, then probabilistically two changes may occur to the probes after they leave the target area,

1. A probe could be deflected by angle θ from the original incoming direction.
2. A probe could acquire a different energy $E \neq E_0$.

Thus, per unit time, there is a certain number of incoming probes dN that are deflected into certain $(\theta, \theta + d\theta)$ that corresponds to solid angle $d\Omega = -2\pi d\cos\theta$ (we use $d\Omega$ instead of $d\theta d\phi$ because there is axial symmetry) and with outgoing kinetic energies between $(E, E + dE)$,

$$dN = f\frac{d^2\sigma}{d\Omega dE}d\Omega dE, \tag{42}$$

and this can be taken as the definition of $d^2\sigma/d\Omega dE$.

a. Assume the target is transfixed (constrained to be immobile), calculate $d^2\sigma/d\Omega dE$. You are allowed to use the δ-function.

b. Assume the target is free and at rest before collision, calculate $d^2\sigma/d\Omega dE$. You are allowed to use the δ-function.

c. Verify that in both cases,

$$\int\int \frac{d^2\sigma}{d\Omega dE}d\Omega dE = 4\pi a^2. \tag{43}$$

Explain why this should be obvious.

Answer:

a. When the target is transfixed, there is no energy transfer so $E = E_0$. As for the momentum transfer, there is a simple relationship between θ and the contact point characterized by the contact angle α,

$$\alpha = \frac{\pi - \theta}{2}. \tag{44}$$

At the contact, the Cartesian coordinate of the probe's center is,

$$(x, y) = (-2a\cos\alpha, \quad 2a\sin\alpha), \tag{45}$$

so there is,

$$y = 2a\sin\left(\frac{\pi - \theta}{2}\right), \tag{46}$$

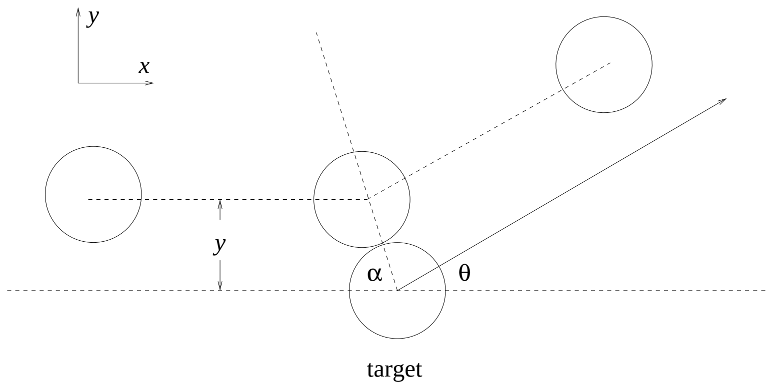

Figure 7: Collision cross-section.

and therefore,

$$|dy| = \left| a \cos\left(\frac{\pi - \theta}{2}\right) d\theta \right|, \tag{47}$$

which says that if the incoming probe happens to be inside the ring $(y, y + dy)$ of thickness dy, the outgoing angle will be between $(\theta, \theta + d\theta)$. That ring has differential cross-section,

$$d\sigma = |2\pi y dy|, \tag{48}$$

to intercept the probe flux, and the corresponding outgoing solid angle is,

$$d\Omega = 2\pi \sin\theta d\theta, \tag{49}$$

therefore,

$$\frac{d\sigma}{d\Omega} = \frac{|2\pi y dy|}{|2\pi \sin\theta d\theta|} = \left| \frac{2a^2 \sin\left(\frac{\pi-\theta}{2}\right) \cos\left(\frac{\pi-\theta}{2}\right)}{\sin\theta} \right| = a^2. \tag{50}$$

The double differential cross-section $d^2\sigma/d\Omega dE$ is formally,

$$\frac{d\sigma}{d\Omega dE} = a^2 \delta(E - E_0), \tag{51}$$

and the total scattering cross-section of hard spheres is simply,

$$\int\int \frac{d\sigma}{d\Omega dE} d\Omega dE = 4\pi a^2. \tag{52}$$

b. When the target is free, there will be both momentum and energy transfer, and the relationship between α and θ is not straightforward. One way of solving $\theta(\alpha)$ is to write down the energy and momentum conservation equations with an additional constraint that the momentum transfer is in the direction of contact, α. A simpler method, however, is by switching into a different inertial frame comoving with speed $v_f = v/2$, where v is the incoming probe speed in rest frame: $E_0 \equiv mv^2/2$. In the comoving frame, the probe comes in with speed $v/2$, and exits with speed $v/2$ in the direction θ'. The target does exactly the opposite. And it is clear that in the comoving frame,

$$\alpha \;=\; \frac{\pi - \theta'}{2}, \quad \theta' \;=\; \pi - 2\alpha, \tag{53}$$

is still valid. The exit velocity of the probe in the comoving frame is therefore,

$$(v'_x, v'_y) \;=\; (v\cos\theta'/2, \;\; v\sin\theta'/2), \tag{54}$$

and transforming back to the rest frame, it is,

$$(v_x, v_y) \;=\; (v_x + v_f, \;\; v_y) \;=\; (v(1+\cos\theta')/2, \;\; v\sin\theta'/2), \tag{55}$$

therefore,

$$\cos\theta \;=\; \frac{1+\cos\theta'}{\sqrt{(1+\cos\theta')^2 + \sin^2\theta'}} \;=\; \frac{1-\cos 2\alpha}{\sqrt{(1-\cos 2\alpha)^2 + \sin^2 2\alpha}} \;=\; \sin\alpha. \tag{56}$$

As α varies from 0 to $\pi/2$, so does θ. There is,

$$\frac{d\sigma}{d\Omega} \;=\; \frac{|\pi dy^2|}{|2\pi d\cos\theta|} \;=\; \left|\frac{4a^2 d\sin^2\alpha}{2d\cos\theta}\right| \;=\; 4a^2\cos\theta, \quad \theta \in (0, \pi/2). \tag{57}$$

The exit energy is,

$$E = \frac{m}{2}(v_x^2 + v_y^2) = \frac{1+\cos\theta'}{2}E_0 = \cos^2\theta E_0, \tag{58}$$

therefore,

$$\frac{d\sigma}{d\Omega dE} \;=\; 4a^2\cos\theta\delta(E - \cos^2\theta E_0), \tag{59}$$

and,

$$\int\!\!\int \frac{d\sigma}{d\Omega dE}d\Omega dE \;=\; \int_0^{\pi/2} 2\pi d\cos\theta(4a^2\cos\theta) \;=\; 4\pi a^2. \tag{60}$$

c. Whenever a probe is within the cylinder of radius $2a$ to the target, its trajectory will be influenced. Therefore the total cross-section must be $\pi(2a)^2 = 4\pi a^2$.

4 Poisson Bracket (25 pt)

Question:

a. Verify Jacobi's identity:

$$\{f,\{g,h\}\} \;+\; \{g,\{h,f\}\} \;+\; \{h,\{f,g\}\} \;\;=\;\; 0. \tag{61}$$

b. Use the above to show why if f and g are constants of motion, then $\{f,g\}$ must also be.

c. Using (2.144), prove that,

$$\{L_x, L_y\} = L_z, \;\; \{L_y, L_z\} = L_x, \;\; \{L_z, L_x\} = L_y. \tag{62}$$

Answer: **a**. Using shorthand notations for partial derivative and index summation,

$$\{g,h\} = g_{q_k}h_{p_k} - h_{q_k}g_{p_k}, \;\; \{h,f\} = h_{q_k}f_{p_k} - f_{q_k}h_{p_k}, \;\; \{f,g\} = f_{q_k}g_{p_k} - g_{q_k}f_{p_k}, \tag{63}$$

$$\{f,\{g,h\}\} = f_{q_l}(g_{q_kp_l}h_{p_k}+g_{q_k}h_{p_kp_l}-h_{q_kp_l}g_{p_k}-h_{q_k}g_{p_kp_l})-f_{p_l}(g_{q_kq_l}h_{p_k}+g_{q_k}h_{p_kq_l}-h_{q_kq_l}g_{p_k}-h_{q_k}g_{p_kq_l}),$$

$$\{g,\{h,f\}\} = g_{q_l}(h_{q_kp_l}f_{p_k}+h_{q_k}f_{p_kp_l}-f_{q_kp_l}h_{p_k}-f_{q_k}h_{p_kp_l})-g_{p_l}(h_{q_kq_l}f_{p_k}+h_{q_k}f_{p_kq_l}-f_{q_kq_l}h_{p_k}-f_{q_k}h_{p_kq_l}),$$

$$\{h,\{f,g\}\} = h_{q_l}(f_{q_kp_l}g_{p_k}+f_{q_k}g_{p_kp_l}-g_{q_kp_l}f_{p_k}-g_{q_k}f_{p_kp_l})-h_{p_l}(f_{q_kq_l}g_{p_k}+f_{q_k}g_{p_kq_l}-g_{q_kq_l}f_{p_k}-g_{q_k}f_{p_kq_l}),$$

They indeed sum up to zero.

b. If f and g are constants of motion, then

$$\{g,\mathcal{H}\} = 0, \;\; \{\mathcal{H},f\} = 0, \tag{64}$$

and from (**a**) there is,

$$\{f,\{g,\mathcal{H}\}\} \;+\; \{g,\{\mathcal{H},f\}\} \;+\; \{\mathcal{H},\{f,g\}\} \;\;=\;\; 0, \tag{65}$$

therefore,

$$\{\mathcal{H},\{f,g\}\} \;\;=\;\; 0, \tag{66}$$

which means $\{f,g\}$ is also a constant of motion.

c. In Levi-Cevita notation,

$$L_i \;\;\equiv\;\; \epsilon_{ijk}r_jp_k, \tag{67}$$

So,

$$
\begin{aligned}
\{L_i, L_{i'}\} &= \{\epsilon_{ijk} r_j p_k, \epsilon_{i'j'k'} r_{j'} p_{k'}\} \\
&= \epsilon_{ijk}\epsilon_{i'j'k'}\{r_j p_k, r_{j'} p_{k'}\} \\
&= \epsilon_{ijk}\epsilon_{i'j'k'}(r_j\{p_k, r_{j'}\}p_{k'} + \{r_j, p_{k'}\}p_k r_{j'}) \\
&= \epsilon_{ijk}\epsilon_{i'j'k'}(-r_j\delta_{kj'}p_{k'} + \delta_{jk'}p_k r_{j'}) \\
&= -\epsilon_{ijk}\epsilon_{i'kk'} r_j p_{k'} + \epsilon_{ijk}\epsilon_{i'j'j} p_k r_{j'} \\
&= (\delta_{ii'}\delta_{jk'} - \delta_{ik'}\delta_{i'j}) r_j p_{k'} - (\delta_{ii'}\delta_{kj'} - \delta_{ij'}\delta_{ki'}) p_k r_{j'} \\
&= \delta_{ii'} r_j p_j - r_{i'} p_i - \delta_{ii'} p_k r_k + p_{i'} r_i \\
&= r_i p_{i'} - r_{i'} p_i.
\end{aligned} \tag{68}
$$

For example, if $i = x$, $j = y$, then,

$$
\{L_x, L_y\} = r_x p_y - r_y p_x = L_z. \tag{69}
$$