22.51 Problem Set 4 (due Fri, Oct. 5)

1 Oscillation (Sturm) Theorem (25 pt)

Question: Suppose function $\psi_1(x)$ satisfies,

$$\psi_1''(x) = \frac{2m}{\hbar^2}(V(x) - E_1)\psi_1(x), \quad x \in (-\infty, \infty), \tag{1}$$

and function $\psi_2(x)$ satisfies,

$$\psi_2''(x) = \frac{2m}{\hbar^2}(V(x) - E_2)\psi_2(x), \quad x \in (-\infty, \infty), \tag{2}$$

where all relevant quantities are real.

(**a**). Prove that,

$$\psi_1'(x)\psi_2(x) - \psi_1(x)\psi_2'(x)\Big|_{x_0}^{x_1} = \frac{2m}{\hbar^2}(E_2 - E_1)\int_{x_0}^{x_1}\psi_1(x)\psi_2(x)dx, \tag{3}$$

for any $x_1 > x_0$.

(**b**). Suppose $E_1 < E_2$, prove that between any two consecutive nodes of $\psi_1(x)$, there exists at least one node of $\psi_2(x)$.

(**c**). When both $\psi_1(x)$ and $\psi_2(x)$ are bound-states and $V(x)$ is smooth, and $E_1 < E_2$, show that there exist extra nodes of $\psi_2(x)$ to the left of the leftmost node of $\psi_1(x)$, and extra nodes to the right of the rightmost node of $\psi_1(x)$. Therefore, $\psi_2(x)$ must have at least one more node than $\psi_1(x)$.

2 Bloch Theorem (25 pt)

Question: The Hamiltonian of a single particle in a 1D periodic potential is,

$$\hat{\mathcal{H}} \equiv \frac{\hat{p}^2}{2m} + V(x), \quad V(x-a) = V(x) \;\; \forall x \in (-\infty, \infty). \tag{4}$$

Let us define translation operator $\hat{T}$,

$$\hat{T}\psi(x) \equiv \psi(x-a). \tag{5}$$

Prove that,

(**a**). Each eigenfunction $\psi(x)$ of $\hat{T}$ must satisfy,

$$\hat{T}\psi(x) \;=\; \exp(-ika)\psi(x), \tag{6}$$

for some $k \in [-\pi/a, \pi/a)$.

(**b**). Rationalize without rigorous proof that we should be able label the eigenfunctions of $\hat{\mathcal{H}}$ by $k \in [-\pi/a, \pi/a)$ and maybe other quantum number(s) n,

$$\hat{T}\psi_k^n(x) \;=\; \exp(-ika)\psi_k^n(x), \quad \hat{\mathcal{H}}\psi_k^n(x) \;=\; E_n(k)\psi_k^n(x), \tag{7}$$

where $E_n(k)$ is called the energy band.

3 Bound-State(s) of δ-Function Potential

Question: Solve for the bound-state(s) of,

$$\hat{\mathcal{H}} \;\equiv\; \frac{\hat{p}^2}{2m} - V_0\delta(x), \quad V_0 > 0. \tag{8}$$

4 Eigenfunctions of comb-Function Potential

Question: Using the conclusion of **2(b)**, solve for the eigenfunctions of,

$$\hat{\mathcal{H}} \;\equiv\; \frac{\hat{p}^2}{2m} - V_0 \sum_{n=-\infty}^{\infty} \delta(x-n), \quad V_0 > 0. \tag{9}$$