III.H Zeroth order hydrodynamics

As a first approximation, we shall assume that in "local equilibrium," the density f_1 at each point in space can be represented as in eq.(III.53), i.e.

$$f_1^0(\vec{p},\vec{q},t) = \frac{n(\vec{q},t)}{\left(2\pi m k_B T(\vec{q},t)\right)^{3/2}} \exp\left[-\frac{\left(\vec{p}-m\vec{u}(\vec{q},t)\right)^2}{2mk_BT(\vec{q},t)}\right]. \tag{III.90}$$

The choice of parameters clearly enforces $\int d^3\vec{p} f_1^0 = n$, and $\langle \vec{p}/m\rangle^0 = \vec{u}$, as required. Average values are easily calculated for the Gaussian form; in particular

$$\langle c_\alpha c_\beta\rangle^0 = \frac{k_BT}{m}\delta_{\alpha\beta}\,, \tag{III.91}$$

leading to

$$P^0_{\alpha\beta} = nk_BT\delta_{\alpha\beta}\,, \qquad \text{and} \qquad \varepsilon = \frac{3}{2}k_BT. \tag{III.92}$$

Since the density f_1^0 is even in $\vec{c}$, all odd expectation values vanish, and in particular

$$\vec{h}^0 = 0. \tag{III.93}$$

The conservation laws in this approximation take the simple forms

$$\begin{cases} D_t n = -n\partial_\alpha u_\alpha \\ mD_t u_\alpha = F_\alpha - \dfrac{1}{n}\partial_\alpha\left(nk_BT\right) \\ D_tT = -\dfrac{2}{3}T\partial_\alpha u_\alpha \end{cases}. \tag{III.94}$$

In the above expression, we have introduced the *material derivative*

$$D_t \equiv \left[\partial_t + u_\beta\partial_\beta\right], \tag{III.95}$$

which measures the time variations of a quantity as it moves along the stream-lines set up by the average velocity field $\vec{u}$. By combining the first and third equations, it is easy to get

$$D_t \ln\left(nT^{-3/2}\right) = 0. \tag{III.96}$$

The quantity $\ln\left(nT^{-3/2}\right)$ is like a local entropy for the gas (see eq.(III.64)), which according to the above equation is not changed along stream-lines. The zeroth order hydrodynamics thus predicts that the gas flow is adiabatic. This prevents the local equilibrium solution of eq.(III.90) from reaching a true global equilibrium form.

To demonstrate that eqs.(III.94) do not describe a satisfactory approach to equilibrium, examine the evolution of small deformations about a stationary ($\vec{u}_0 = 0$) state, in a uniform box ($\vec{F} = 0$), by setting

$$\begin{cases} n(\vec{q}, t) = \overline{n} + \nu(\vec{q}, t) \\ T(\vec{q}, t) = \overline{T} + \theta(\vec{q}, t) \end{cases} . \tag{III.97}$$

We shall next expand eqs.(III.94) to *first order* in the deviations $(\nu, \theta, \vec{u})$. Note that to lowest order, $D_t = \partial_t + O(u)$, leading to the linearized zeroth order hydrodynamic equations

$$\begin{cases} \partial_t \nu = -\overline{n} \partial_\alpha u_\alpha \\ m \partial_t u_\alpha = -\dfrac{k_B \overline{T}}{\overline{n}} \partial_\alpha \nu - k_B \partial_\alpha \theta \\ \partial_t \theta = -\dfrac{2}{3} \overline{T} \partial_\alpha u_\alpha \end{cases} . \tag{III.98}$$

- *Normal modes* of the system are obtained by Fourier transformations,

$$A\left(\vec{k}, \omega\right) = \int d^3q dt \exp\left[i\left(\vec{k} \cdot \vec{q} - \omega t\right)\right] A\left(\vec{q}, t\right), \tag{III.99}$$

where A stands for any of $(\nu, \theta, \vec{u})$. The natural vibration frequencies are solutions to the matrix equation

$$\omega \begin{pmatrix} \nu \\ u_\alpha \\ \theta \end{pmatrix} = \begin{pmatrix} 0 & \overline{n} \delta_{\alpha\beta} k_\beta & 0 \\ \frac{k_B \overline{T}}{m \overline{n}} \delta_{\alpha\beta} k_\beta & 0 & \frac{k_B}{m} \delta_{\alpha\beta} k_\beta \\ 0 & \frac{2}{3} \overline{T} \delta_{\alpha\beta} k_\beta & 0 \end{pmatrix} \begin{pmatrix} \nu \\ u_\beta \\ \theta \end{pmatrix} . \tag{III.100}$$

It is easy to check that this equation has three zero frequency modes. One corresponds to "entropy" waves, as noted before, satisfying $\nu/\overline{n} + 3\theta/2\overline{T} = 0$. There is also no evolution for *transverse* velocity modes which satisfy $\vec{k} \cdot \vec{u}_T = 0$, i.e. shear flows are not relaxed in the zeroth order approximation. Finally, the *longitudinal* velocity ($\vec{u}_\ell \parallel \vec{k}$) combines with density and temperature fluctuations in eigenmodes of the form

$$\begin{pmatrix} \overline{n} |\vec{k}| \\ \omega(\vec{k}) \\ \frac{2}{3} \overline{T} |\vec{k}| \end{pmatrix}, \qquad \text{with} \qquad \omega(\vec{k}) = \pm v_\ell |\vec{k}|, \tag{III.101}$$

where

$$v_\ell = \sqrt{\frac{5}{3} \frac{k_B \overline{T}}{m}}, \tag{III.102}$$

is the longitudinal sound velocity.

We thus find that none of the conserved quantities relaxes to equilibrium in the zeroth order approximation. Shear flow and entropy modes persist forever, while the two sound modes have undamped oscillations. This is a deficiency of the zeroth order hydrodynamics, which is removed by finding a better solution to the Boltzmann equation.

III.I First order hydrodynamics

While $f_1^0(\vec{p},\vec{q},t)$ of eq.(III.90) does set the right hand side of the Boltzmann equation to zero, it is not a full solution, as the left hand side causes its form to vary. The left hand side is a *linear* differential operator, which using the various notations introduced in the previous sections, can be written as

$$\mathcal{L}\left[f\right] \equiv \left[\partial_t + \frac{p_\alpha}{m}\partial_\alpha + F_\alpha \frac{\partial}{\partial p_\alpha}\right] f = \left[D_t + c_\alpha \partial_\alpha + \frac{F_\alpha}{m}\frac{\partial}{\partial c_\alpha}\right] f. \tag{III.103}$$

It is simpler to examine the effect of $\mathcal{L}$ on $\ln f_1^0$. which can be written as

$$\ln f_1^0 = \ln\left(nT^{-3/2}\right) - \frac{mc^2}{2k_BT} - \frac{3}{2}\ln\left(2\pi m k_B\right). \tag{III.104}$$

Using the relation $\partial(c^2/2) = c_\beta \partial c_\beta = -c_\beta \partial u_\beta$, we get

$$\begin{aligned}\mathcal{L}\left[\ln f_1^0\right] =& D_t \ln\left(nT^{-3/2}\right) + \frac{mc^2}{2k_BT^2}D_tT + \frac{m}{k_BT}c_\alpha D_t u_\alpha \\ &+ c_\alpha\left(\frac{\partial_\alpha n}{n} - \frac{3}{2}\frac{\partial_\alpha T}{T}\right) + \frac{mc^2}{2k_BT^2}c_\alpha \partial_\alpha T + \frac{m}{k_BT}c_\alpha c_\beta \partial_\alpha u_\beta - \frac{F_\alpha c_\alpha}{k_BT}.\end{aligned} \tag{III.105}$$

If the quantities n, T, and u_α, satisfy the zeroth order hydrodynamic eqs.(III.94), we can simplify the above equation to

$$\begin{aligned}\mathcal{L}\left[\ln f_1^0\right] =& 0 - \frac{mc^2}{3k_BT^2}\partial_\alpha u_\alpha + c_\alpha\left[\left(\frac{F_\alpha}{k_BT} - \frac{\partial_\alpha n}{n} - \frac{\partial_\alpha T}{T}\right) + \left(\frac{\partial_\alpha n}{n} - \frac{3}{2}\frac{\partial_\alpha T}{T}\right) - \frac{F_\alpha}{k_BT}\right] \\ &+ \frac{mc^2}{2k_BT^2}c_\alpha \partial_\alpha T + \frac{m}{k_BT}c_\alpha c_\beta u_{\alpha\beta} \\ =& \frac{m}{k_BT}\left(c_\alpha c_\beta - \frac{\delta_{\alpha\beta}}{3}c^2\right)u_{\alpha\beta} + \left(\frac{mc^2}{2k_BT} - \frac{5}{2}\right)\frac{c_\alpha}{T}\partial_\alpha T.\end{aligned} \tag{III.106}$$

The characteristic time scale τ_U for $\mathcal{L}$ is *extrinsic*, and can be made much larger than $\tau_\times$. The zeroth order result is thus exact in the limit $(\tau_\times/\tau_U) \to 0$; and corrections can be constructed in a perturbation series in $(\tau_\times/\tau_U)$. To this purpose, we set $f_1 = f_1^0(1+g)$, and linearize the collision operator as

$$\begin{aligned}C\left[f_1, f_1\right] =& -\int d^3\vec{p}_2 d^2\vec{b}|\vec{v}_1 - \vec{v}_2| f_1^0(\vec{p}_1) f_1^0(\vec{p}_2)\left[g(\vec{p}_1) + g(\vec{p}_2) - g(\vec{p}_1{}') - g(\vec{p}_2{}')\right] \\ \equiv& - f_1^0(\vec{p}_1) C_L[g].\end{aligned} \tag{III.107}$$

While linear, the above integral operator is still difficult to manipulate in general. As a first approximation, and noting its characteristic magnitude, we set

$$C_L[g] \approx \frac{g}{\tau_\times}. \tag{III.108}$$

This is known as the *single collision time* approximation, and from the linearized Boltzmann equation $\mathcal{L}[f_1] = -f_1^0 C_L[g]$, we obtain

$$g = -\tau_\times \frac{1}{f_1^0}\mathcal{L}\left[f_1\right] \approx -\tau_\times \mathcal{L}\left[\ln f_1^0\right], \tag{III.109}$$

where we have kept only the leading term. Thus the first order solution is given by (using eq.(III.106))

$$f_1^1(\vec{p},\vec{q},t) = f_1^0(\vec{p},\vec{q},t)\left[1 - \frac{\tau_\mu m}{k_B T}\left(c_\alpha c_\beta - \frac{\delta_{\alpha\beta}}{3}c^2\right)u_{\alpha\beta} - \tau_K\left(\frac{mc^2}{2k_BT} - \frac{5}{2}\right)\frac{c_\alpha}{T}\partial_\alpha T\right], \tag{III.110}$$

where $\tau_\mu = \tau_K = \tau_\times$ in the single collision time approximation. However, in writing the above equation, we have anticipated the possibility of $\tau_\mu \neq \tau_K$ which arises in more sophisticated treatments (although both times are still of order of $\tau_\times$).

It is easy to check that $\int d^3\vec{p} f_1^1 = \int d^3\vec{p} f_1^0 = n$, and thus various local expectation values are calculated to first order as

$$\langle \mathcal{O} \rangle^1 = \frac{1}{n}\int d^3\vec{p}\,\mathcal{O} f_1^0(1+g) = \langle \mathcal{O} \rangle^0 + \langle g\mathcal{O} \rangle^0 + \cdots. \tag{III.111}$$

The calculation of averages over products of c_α's, distributed according to the Gaussian weight of f_1^0, is greatly simplified by the use of *Wick's theorem*, which states that expectation value of the product is the sum over all possible products of paired expectation values, for example

$$\langle c_\alpha c_\beta c_\gamma c_\delta \rangle_0 = \left(\frac{k_BT}{m}\right)^2 \left(\delta_{\alpha\beta}\delta_{\gamma\delta} + \delta_{\alpha\gamma}\delta_{\beta\delta} + \delta_{\alpha\delta}\delta_{\beta\gamma}\right). \tag{III.112}$$

(Expectation values involving a product of an odd number of c_α's are zero by symmetry.) Using this result, it is easy to verify that

$$\left\langle \frac{p_\alpha}{m} \right\rangle^1 = u_\alpha - \tau_K \frac{\partial_\beta T}{T}\left\langle \left(\frac{mc^2}{2k_BT} - \frac{5}{2}\right) c_\alpha c_\beta \right\rangle = u_\alpha. \tag{III.113}$$

The pressure tensor at first order is given by

$$\begin{aligned}P^1_{\alpha\beta} =& nm\left\langle c_\alpha c_\beta\right\rangle^1 = nm\left[\left\langle c_\alpha c_\beta\right\rangle^0 - \frac{\tau_\mu m}{k_BT}\left\langle c_\alpha c_\beta\left(c_\mu c_\nu - \frac{\delta_{\mu\nu}}{3}c^2\right)\right\rangle^0\right]\\ =& nk_BT\delta_{\alpha\beta} - 2nk_BT\tau_\mu\left(u_{\alpha\beta} - \frac{\delta_{\alpha\beta}u_{\gamma\gamma}}{3}\right).\end{aligned} \tag{III.114}$$

(Using the above result, we can further verify that $\varepsilon^1 = \left\langle mc^2/2\right\rangle^1 = 3k_BT/2$, as before.) Finally, the heat flux is given by

$$\begin{aligned}h^1_\alpha =& n\left\langle c_\alpha\frac{mc^2}{2}\right\rangle^1 = -\frac{nm\tau_K}{2}\frac{\partial_\beta T}{T}\left\langle\left(\frac{mc^2}{2k_BT} - \frac{5}{2}\right)c_\alpha c_\beta c^2\right\rangle^0\\ =& -\frac{5}{2}\frac{nk_B^2T\tau_K}{m}\partial_\alpha T.\end{aligned} \tag{III.115}$$

At this order, we find that spatial variations in temperatures generate a heat flow that tends to smooth them out, while shear flows are opposed by the off-diagonal terms in the pressure tensor. These effects are sufficient to cause relaxation to equilibrium, as can be seen by examining the linearized hydrodynamic equations. There is now a contribution to $D_tu_\alpha \approx \partial_t u_\alpha$, given by

$$\delta^1\left(\partial_t u_\alpha\right) \equiv \frac{1}{mn}\partial_\beta\delta^1P_{\alpha\beta} \approx -\frac{\mu}{m\overline{n}}\left(\frac{1}{3}\partial_\alpha\partial_\beta + \delta_{\alpha\beta}\partial_\gamma\partial_\gamma\right)u_\beta, \tag{III.116}$$

where we have introduced the *viscosity coefficient* $\mu \equiv k_B\overline{T}\overline{n}\tau_\mu$. Similarly, there is a first order correction to the equation for $D_tT \approx \partial_t\theta$, which is given by

$$\delta^1\left(\partial_t\theta\right) \equiv -\frac{2}{3k_Bn}\partial_\alpha h_\alpha \approx -\frac{2K}{3k_B\overline{n}}\partial_\alpha\partial_\alpha\theta, \tag{III.117}$$

where $K = (5k_B^2\overline{T}\overline{n}\tau_K)/(2m)$ is the coefficient of *thermal conductivity* of the gas.

After Fourier transformation, the matrix equation (III.100) is modified to

$$\omega\begin{pmatrix}\nu\\ u_\alpha\\ \theta\end{pmatrix} = \begin{pmatrix}0 & \overline{n}\delta_{\alpha\beta}k_\beta & 0\\ \frac{k_B\overline{T}}{m\overline{n}}\delta_{\alpha\beta}k_\beta & -i\frac{\mu}{m\overline{n}}\left(k^2\delta_{\alpha\beta} + \frac{k_\alpha k_\beta}{3}\right) & \frac{k_B}{m}\delta_{\alpha\beta}k_\beta\\ 0 & \frac{2}{3}\overline{T}\delta_{\alpha\beta}k_\beta & -i\frac{2Kk^2}{3k_B\overline{n}}\end{pmatrix}\begin{pmatrix}\nu\\ u_\beta\\ \theta\end{pmatrix}. \tag{III.118}$$

It is simple to verify that the longitudinal normal models ($\vec{k}\cdot\vec{u}_T = 0$) have a frequency

$$\omega_T = -i\frac{\mu}{m\overline{n}}k^2. \tag{III.119}$$

The imaginary frequency implies that these modes are damped over a characteristic time $\tau_T(k) \sim 1/|\omega_T| \sim (\lambda)^2/(\tau_\mu\overline{v}^2)$, where λ is the corresponding wavelength, and $\overline{v} \sim \sqrt{k_BT/m}$ is a typical gas particle velocity. We see that the characteristic time scales grow as the square of the wavelength, which is characteristic of *diffusive* processes. Similarly, the entropy mode become diffusive, while the longitudinal sound modes turn into damped oscillations. It is this damping that guarantees the, albeit slow, approach of the gas to its final uniform and stationary equilibrium state.