Quantum Phenomena

1. *One dimensional chain:* A chain of $N+1$ particles of mass m is connected by N massless springs of spring constant K and relaxed length a. The first and last particles are held fixed at the equilibrium separation of Na. Let us denote the longitudinal displacements of the particles from their equilibrium positions by $\{u_i\}$, with $u_0 = u_N = 0$ since the end particles are fixed. The Hamiltonian governing $\{u_i\}$, and the conjugate momenta $\{p_i\}$, is

$$\mathcal{H} = \sum_{i=1}^{N-1} \frac{p_i^2}{2m} + \frac{K}{2}\left[u_1^2 + \sum_{i=1}^{N-2}\left(u_{i+1} - u_i\right)^2 + u_{N-1}^2\right].$$

(a) Using the appropriate (sine) Fourier transforms, find the normal modes $\{\tilde{u}_k\}$, and the corresponding frequencies $\{\omega_k\}$.

(b) Express the Hamiltonian in terms of the amplitudes of normal modes $\{\tilde{u}_k\}$, and evaluate the *classical* partition function. (You may integrate the $\{u_i\}$ from $-\infty$ to $+\infty$).

(c) First evaluate $\left\langle |\tilde{u}_k|^2 \right\rangle$, and use the result to calculate $\left\langle u_i^2 \right\rangle$. Plot the resulting squared displacement of each particle as a function of its equilibrium position.

(d) How are the results modified if only the first particle is fixed ($u_0 = 0$), while the other end is free ($u_N \neq 0$)? (Note that this is a much simpler problem as the partition function can be evaluated by changing variables to the $N-1$ spring extensions.)

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2. *Black Hole Thermodynamics:* According to Bekenstein and Hawking, the entropy of a black hole is proportional to its area A, and given by

$$S = \frac{k_B c^3}{4G\hbar} A \quad .$$

(a) Calculate the escape velocity at a radius R from a mass M using classical mechanics. Find the relationship between the radius and mass of a black hole by setting this escape velocity to the speed of light c. (Relativistic calculations do not modify this result, originally obtained by Laplace.)

(b) Does entropy increase or decrease when two black holes collapse into one? What is the entropy change for the universe (in equivalent number of bits of information), when two solar mass black holes ($M_\odot \approx 2 \times 10^{30} kg$) coalesce?

(c) The internal energy of the black hole is given by the Einstein relation, $E = Mc^2$. Find the temperature of the black hole in terms of its mass.

(d) A "black hole" actually emits thermal radiation due to pair creation processes on its event horizon. Find the rate of energy loss due to such radiation.

(e) Find the amount of time it takes an isolated black hole to evaporate. How long is this time for a black hole of solar mass?

(f) What is the mass of a black hole that is in thermal equilibrium with the current cosmic background radiation at $T = 2.7°K$?

(g) Consider a spherical volume of space of radius R. According to the recently formulated *Holographic Principle* there is a maximum to the amount of entropy that this volume of space can have, independent of its contents! What is this maximal entropy?

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3. *Quantum Oscillator:* Consider a single harmonic oscillator with the Hamiltonian

$$\mathcal{H} = \frac{p^2}{2m} + \frac{m\omega^2 q^2}{2}, \quad \text{with} \quad p = \frac{\hbar}{i}\frac{d}{dq} \quad .$$

(a) Find the partition function Z, at a temperature T, and calculate the energy $\langle \mathcal{H} \rangle$.

(b) Write down the formal expression for the canonical density matrix ρ in terms of the eigenstates $(\{|n>\})$, and energy levels $(\{\epsilon_n\})$ of $\mathcal{H}$.

(c) Show that for a general operator $A(x)$,

$$\frac{\partial}{\partial x} \exp\left[A(x)\right] \neq \frac{\partial A}{\partial x} \exp\left[A(x)\right], \qquad \text{unless} \qquad \left[A, \frac{\partial A}{\partial x}\right] = 0,$$

while in all cases

$$\frac{\partial}{\partial x} \operatorname{tr}\left\{\exp\left[A(x)\right]\right\} = \operatorname{tr}\left\{\frac{\partial A}{\partial x} \exp\left[A(x)\right]\right\}.$$

(d) Note that the partition function calculated in part (a) does not depend on the mass m, i.e. $\partial Z/\partial m = 0$. Use this information, along with the result in part (c), to show that

$$\left\langle \frac{p^2}{2m} \right\rangle = \left\langle \frac{m\omega^2 q^2}{2} \right\rangle .$$

(e) Using the results in parts (d) and (a), or otherwise, calculate $\langle q^2 \rangle$. How are the results in problem 1 modified at low temperatures by inclusion of quantum mechanical effects.

(f) In a coordinate representation, calculate $\langle q'|\rho|q \rangle$ in the high temperature limit. One approach is to use the result

$$\exp(\beta A)\exp(\beta B) = \exp\left[\beta(A+B) + \beta^2[A,B]/2 + \mathcal{O}(\beta^3)\right].$$

(g) At low temperatures, ρ is dominated by low energy states. Use the ground state wave-function to evaluate the limiting behavior of $\langle q'|\rho|q \rangle$ as $T \to 0$.

(h) **(Optional):** Calculate the exact expression for $\langle q'|\rho|q \rangle$.

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Suggested reading: Huang, Chapter 12; Pathria, Chapter 7; R. Penrose, *The Emperor's New Mind*, chapter 7.