8.333: Statistical Mechanics I   Problem Set # **11**   Due: **12/5/03**

Identical Quantum Particles

1. *Particle pair:* Let $Z_1(m)$ denote the partition function for a single quantum particle of mass m in a volume V.

(a) Calculate the partition function of two such particles, if they are bosons, and also if they are (spinless) fermions.

(b) Use the classical approximation $Z_1(m) = V/\lambda^3$ with $\lambda = h/\sqrt{2\pi m k_B T}$. Calculate the corrections to the energy E, and the heat capacity C, due to Bose or Fermi statistics.

(c) At what temperature does the approximation used in (b) break down?

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2. *Solar interior:* According to astrophysical data, the plasma at the center of the sun has the following properties:

Temperature:	$T = 1.6 \times 10^7\,^\circ\mathrm{K}$
Hydrogen density:	$\rho_H = 6 \times 10^4$ kg m^{-3}
Helium density:	$\rho_{He} = 1 \times 10^5$ kg m^{-3}.

(a) Obtain the thermal wavelengths for electrons, protons, and α-particles (nuclei of He).

(b) Assuming that the gas is ideal, determine whether the electron, proton, or α-particle gases are degenerate in the quantum mechanical sense.

(c) Estimate the total gas pressure due to these gas particles near the center of the sun.

(d) Estimate the total radiation pressure close to the center of the sun. Is it matter, or radiation pressure, that prevents the gravitational collapse of the sun?

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3. *Anharmonic dispersion:* Consider a gas of non-interacting identical (spinless) quantum particles with an energy spectrum $\varepsilon = |\vec{p}/\hbar|^s$, contained in a box of "volume" V in d-dimensions.

(a) Calculate the grand potential $\mathcal{G}_\eta = -k_B T \ln \mathcal{Q}_\eta$, and the density $n = N/V$, at a chemical potential μ. Express your answers in terms of s, d, and $f_m^\eta(z)$, where $z = e^{\beta\mu}$, and

$$f_m^\eta(z) = \frac{1}{\Gamma(m)} \int_0^\infty \frac{dx x^{m-1}}{z^{-1}e^x - \eta}.$$

(Hint: Use integration by parts on the expression for $\ln \mathcal{Q}_\eta$.)

(b) Find the ratio PV/E, and compare it to the classical result obtained in problem set 7.

(c) *For fermions*, calculate the dependence of E/N, and P, on the density $n = N/V$, at zero temperature. (Hint: $f_m(z) \to (\ln z)^m/m!$ as $z \to \infty$.)

(d) **(Optional)** *For bosons*, find the dimension $d_\ell(s)$, below which there is no Bose condensation. Is there condensation for $s = 2$ at $d = 2$?

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4. *Pauli Paramagnetism:* Calculate the contribution of electron spin to its magnetic susceptibility as follows. Consider non-interacting electrons, each subject to a Hamiltonian

$$\mathcal{H}_1 = \frac{\vec{p}^{\,2}}{2m} - \mu_0 \vec{\sigma} \cdot \vec{B} \quad ,$$

where $\mu_0 = e\hbar/2mc$, and the eigenvalues of $\vec{\sigma} \cdot \vec{B}$ are $\pm B$.
(The orbital effect, $\vec{p} \to \vec{p} - e\vec{A}$, has been ignored.)

(a) Calculate the grand potential $\mathcal{G}_- = -k_B T \ln \mathcal{Q}_-$, at a chemical potential μ.

(b) Calculate the densities $n_+ = N_+/V$, and $n_- = N_-/V$, of electrons pointing parallel and antiparallel to the field.

(c) Obtain the expression for the magnetization $M = \mu_0(N_+ - N_-)$, and expand the result for small B.

(d) Sketch the zero field susceptibility $\chi(T) = \partial M/\partial B|_{B=0}$, and indicate its behavior at low and high temperatures.

(e) Estimate the magnitude of χ/N for a typical metal at room temperature.

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5. (Optional)

(a) Give numerical estimates of the Fermi energy and Fermi temperature for (i) Electrons in a typical metal; (ii) Nucleons in a heavy nucleus; (iii) He^3 atoms in liquid He^3 (atomic volume $= 46.2\text{Å}^3$ per atom).

(b) Estimate the ratio of the electron and phonon heat capacities at room temperature for a typical metal.

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Suggested Reading: Huang, Chapters 11–13; Ma, Chapters 4, 16.