Equilibrium in Kinetic Theory

1. A thermalized gas particle is suddenly confined to a one–dimensional trap. The corresponding mixed state is described by an initial density function $\rho(q, p, t = 0) = \delta(q)f(p)$, where $f(p) = \exp(-p^2/2mk_BT)/\sqrt{2\pi mk_BT}$.

(a) Starting from Liouville's equation, derive $\rho(q, p, t)$ and sketch it in the (q, p) plane.

(b) Derive the expressions for the averages $\left\langle q^2\right\rangle$ and $\left\langle p^2\right\rangle$ at $t > 0$.

(c) Suppose that hard walls are placed at $q = \pm Q$. Describe $\rho(q, p, t \gg \tau)$, where τ is an appropriately large relaxation time.

(d) A "coarse–grained" density $\tilde{\rho}$, is obtained by ignoring variations of ρ below some small resolution in the (q, p) plane; e.g., by averaging ρ over cells of the resolution area. Find $\tilde{\rho}(q, p)$ for the situation in part (c), and show that it is stationary.

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2. The normalized ensemble density is a probability in the phase space Γ. This probability has an associated entropy $S(t) = -\int d\Gamma \rho(\Gamma, t) \ln \rho(\Gamma, t)$.

(a) Show that if $\rho(\Gamma, t)$ satisfies Liouville's equation for a Hamiltonian $\mathcal{H}$, $dS/dt = 0$.

(b) Using the method of Lagrange multipliers, find the function $\rho_{\max}(\Gamma)$ which maximizes the functional $S[\rho]$, subject to the constraint of fixed average energy, $\langle\mathcal{H}\rangle = \int d\Gamma \rho\mathcal{H} = E$.

(c) Show that the solution to part (b) is stationary, i.e. $\partial\rho_{\max}/\partial t = 0$.

(d) How can one reconcile the result in (a), with the observed increase in entropy as the system approaches the equilibrium density in (b)? (Hint: Think of the situation in 1(d).)

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3. *The Vlasov equation* is obtained in the limit of high particle density $n = N/V$, or large inter-particle interaction range λ, such that $n\lambda^3 \gg 1$. In this limit, the collision terms are dropped from the left hand side of the equations in the BBGKY hierarchy.

(a) Assume that the N body density is a product of one particle densities, i.e. $\rho = \prod_{i=1}^{N} \rho_1(\mathbf{x}_i, t)$, where $\mathbf{x}_i \equiv (\vec{p}_i, \vec{q}_i)$. Calculate the densities f_s, and their normalizations.

(b) Show that once the collision terms are eliminated, all the equations in the BBGKY hierarchy are equivalent to the single equation

$$\left[\frac{\partial}{\partial t} + \frac{\vec{p}}{m}\cdot\frac{\partial}{\partial\vec{q}} - \frac{\partial U_{\text{eff}}}{\partial\vec{q}}\cdot\frac{\partial}{\partial\vec{p}}\right] f_1(\vec{p}, \vec{q}, t) = 0,$$

where

$$U_{\text{eff}}(\vec{q}, t) = U(\vec{q}) + \int d\mathbf{x}'\mathcal{V}(\vec{q} - \vec{q}\,')f_1(\mathbf{x}', t).$$

(c) Now consider N particles confined to a box of volume V, with no additional potential. Show that $f_1(\vec{q}, \vec{p}) = g(\vec{p})/V$ is a stationary solution to the Vlasov equation *for any* $g(\vec{p})$. Why is there no relaxation towards equilibrium for $g(\vec{p})$?

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4. *Two component plasma:* Consider a *neutral* mixture of N ions of charge $+e$ and mass m_+, and N electrons of charge $-e$ and mass m_-, in a volume $V = N/n_0$.

(a) Show that the Vlasov equations for this two component system are

$$\begin{cases} \left[\frac{\partial}{\partial t} + \frac{\vec{p}}{m_+} \cdot \frac{\partial}{\partial \vec{q}} + e\frac{\partial \Phi_{\text{eff}}}{\partial \vec{q}} \cdot \frac{\partial}{\partial \vec{p}}\right] f_+(\vec{p}, \vec{q}, t) = 0 \\ \left[\frac{\partial}{\partial t} + \frac{\vec{p}}{m_-} \cdot \frac{\partial}{\partial \vec{q}} - e\frac{\partial \Phi_{\text{eff}}}{\partial \vec{q}} \cdot \frac{\partial}{\partial \vec{p}}\right] f_-(\vec{p}, \vec{q}, t) = 0 \end{cases},$$

where the effective Coulomb potential is given by

$$\Phi_{\text{eff}}(\vec{q}, t) = \Phi_{\text{ext}}(\vec{q}) + e\int d\mathbf{x}' C(\vec{q} - \vec{q}') \left[f_+(\mathbf{x}', t) - f_-(\mathbf{x}', t)\right].$$

Here, Φ_{ext} is the potential set up by the external charges, and the Coulomb potential $C(\vec{q})$ satisfies the differential equation $\nabla^2 C = 4\pi\delta^3(\vec{q})$.

(b) Assume that the one particle densities have the stationary forms $f_\pm = g_\pm(\vec{p})n_\pm(\vec{q})$. Show that the effective potential satisfies the equation

$$\nabla^2\Phi_{\text{eff}} = 4\pi\rho_{\text{ext}} + 4\pi e\left(n_+(\vec{q}) - n_-(\vec{q})\right),$$

where ρ_{ext} is the external charge density.

(c) Further assuming that the densities relax to the equilibrium Boltzmann weights $n_\pm(\vec{q}) = n_0 \exp\left[\pm\beta e\Phi_{\text{eff}}(\vec{q})\right]$, leads to the self-consistency condition

$$\nabla^2\Phi_{\text{eff}} = 4\pi\left[\rho_{\text{ext}} + n_0 e\left(e^{\beta e\Phi_{\text{eff}}} - e^{-\beta e\Phi_{\text{eff}}}\right)\right],$$

known as the *Poisson–Boltzmann* equation. Due to its nonlinear form, it is generally not possible to solve the Poisson–Boltzmann equation. By linearizing the exponentials, one obtains the simpler *Debye* equation

$$\nabla^2\Phi_{\text{eff}} = 4\pi\rho_{\text{ext}} + \Phi_{\text{eff}}/\lambda^2.$$

Give the expression for the *Debye screening length* λ.

(d) Show that the Debye equation has the general solution

$$\Phi_{\text{eff}}(\vec{q}) = \int d^3\vec{q}' G(\vec{q} - \vec{q}')\rho_{\text{ext}}(\vec{q}'),$$

where $G(\vec{q}) = \exp(-|\vec{q}|/\lambda)/|\vec{q}|$ is the screened Coulomb potential.

(e) Give the condition for the self-consistency of the Vlasov approximation, and interpret it in terms of the inter-particle spacing?

(f) Show that the characteristic relaxation time ($\tau \approx \lambda/c$) is temperature independent. What property of the plasma is it related to?