Two–dimensional electron gas

When donor atoms (such as P or As) are added to a semiconductor (e.g. Si or Ge), their conduction electrons can be thermally excited to move freely in the host lattice. By growing layers of different materials, it is possible to generate a spatially varying potential (work–function) which traps electrons at the boundaries between layers. In the following, we shall treat the trapped electrons as a gas of classical particles *in two dimensions.*

1. *Electron gas in a magnetic field:* If the layer of electrons is sufficiently separated from the donors, the main source of scattering is from electron–electron collisions.

(a) The Hamiltonian for non–interacting free electrons in a magnetic field has the form

$$\mathcal{H}=\sum_i\left[\frac{\left(\vec{p}_i-e\vec{A}\right)^2}{2m}\pm\mu_B|\vec{B}|\right].$$

(The two signs correspond to electron spins parallel or anti-parallel to the field.) The vector potential $\vec{A}=\vec{B}\times\vec{q}/2$ describes a uniform magnetic field $\vec{B}$. Obtain the classical equations of motion, and show that they describe rotation of electrons in cyclotron orbits in a plane orthogonal to $\vec{B}$.

(b) Write down heuristically (i.e. not through a step by step derivation), the Boltzmann equations for the densities $f_\uparrow(\vec{p},\vec{q},t)$ and $f_\downarrow(\vec{p},\vec{q},t)$ of electrons with up and down spins, in terms of the two cross-sections $\sigma\equiv\sigma_{\uparrow\uparrow}=\sigma_{\downarrow\downarrow}$, and $\sigma_\times\equiv\sigma_{\uparrow\downarrow}$, of *spin conserving* collisions.

(c) Show that $d\mathrm{H}/dt\le 0$, where $\mathrm{H}=\mathrm{H}_\uparrow+\mathrm{H}_\downarrow$ is the sum of the corresponding H functions.

(d) Show that $d\mathrm{H}/dt=0$ for any $\ln f$ which is, *at each location*, a linear combination of quantities conserved in the collisions.

(e) Show that the streaming terms in the Boltzmann equation are zero for any function that depends only on the quantities conserved by the one body Hamiltonians.

(f) Show that angular momentum $\vec{L}=\vec{q}\times\vec{p}$, is conserved during, and away from collisions.

(g) Write down the most general form for the equilibrium distribution functions for particles confined to a circularly symmetric potential.

(h) How is the result in part (g) modified by including scattering from magnetic and non-magnetic impurities?

(i) Do conservation of spin and angular momentum lead to new hydrodynamic equations?

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2. *The Lorentz gas* describes non-interacting particles colliding with a fixed set of scatterers. It is a good model for scattering of electrons from donor impurities. Consider a uniform two dimensional density n_0 of fixed impurities, which are hard circles of radius a.

(a) Show that the differential cross section of a hard circle scattering through an angle θ is

$$d\sigma = \frac{a}{2}\,\sin\frac{\theta}{2}\,d\theta,$$

and calculate the total cross section.

(b) Write down the Boltzmann equation for the one particle density $f(\vec{q}, \vec{p}, t)$ of the Lorentz gas (including only collisions with the fixed impurities). *(Ignore the electron spin.)*

(c) Using the definitions $\vec{F} \equiv -\partial U/\partial \vec{q}$, and

$$n(\vec{q}, t) = \int d^2\vec{p} f(\vec{q}, \vec{p}, t), \qquad \text{and} \qquad \langle g(\vec{q}, t)\rangle = \frac{1}{n(\vec{q}, t)} \int d^2\vec{p} f(\vec{q}, \vec{p}, t) g(\vec{q}, t),$$

show that for any function $\chi(|\vec{p}|)$, we have

$$\frac{\partial}{\partial t}\left(n\left\langle \chi\right\rangle\right) + \frac{\partial}{\partial \vec{q}} \cdot \left(n\left\langle \frac{\vec{p}}{m}\chi\right\rangle\right) = \vec{F}\cdot\left(n\left\langle \frac{\partial \chi}{\partial \vec{p}}\right\rangle\right).$$

(d) Derive the conservation equation for local density $\rho \equiv mn(\vec{q}, t)$, in terms of the local velocity $\vec{u} \equiv \langle \vec{p}/m\rangle$.

(e) Since the magnitude of particle momentum is unchanged by impurity scattering, the Lorentz gas has an infinity of conserved quantities $|\vec{p}|^m$. This unrealistic feature is removed upon inclusion of particle–particle collisions. For the rest of this problem focus only on $p^2/2m$ as a conserved quantity. Derive the conservation equation for the energy density

$$\epsilon(\vec{q}, t) \equiv \frac{\rho}{2}\left\langle c^2\right\rangle, \qquad \text{where} \qquad \vec{c} \equiv \frac{\vec{p}}{m} - \vec{u},$$

in terms of the energy flux $\vec{h} \equiv \rho\left\langle \vec{c}c^2\right\rangle/2$, and the pressure tensor $P_{\alpha\beta} \equiv \rho\left\langle c_\alpha c_\beta\right\rangle$.

(f) Starting with a one particle density

$$f^0(\vec{p}, \vec{q}, t) = n(\vec{q}, t)\exp\left[-\frac{p^2}{2mk_BT(\vec{q}, t)}\right]\frac{1}{2\pi mk_BT(\vec{q}, t)},$$

reflecting local equilibrium conditions, calculate $\vec{u}$, $\vec{h}$, and $P_{\alpha\beta}$. Hence obtain the zeroth order hydrodynamic equations.

(g) Show that in the single collision time approximation to the collision term in the Boltzmann equation, the first order solution is

$$f^1(\vec{p}, \vec{q}, t) = f^0(\vec{p}, \vec{q}, t)\left[1 - \tau\frac{\vec{p}}{m}\cdot\left(\frac{\partial \ln\rho}{\partial \vec{q}} - \frac{\partial \ln T}{\partial \vec{q}} + \frac{p^2}{2mk_BT^2}\frac{\partial T}{\partial \vec{q}} - \frac{\vec{F}}{k_BT}\right)\right].$$

(h) Show that using the first order expression for f, we obtain

$$\rho\vec{u} = n\tau\left[\vec{F} - k_BT\nabla\ln\left(\rho T\right)\right].$$

(i) From the above equation, calculate the velocity response function $\chi_{\alpha\beta} = \partial u_\alpha/\partial F_\beta$.

(j) **(Optional)** Calculate $P_{\alpha\beta}$, and $\vec{h}$, and hence write down the first order hydrodynamic equations.

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