Identical Quantum Particles

1. *Particle pair:*

(a) *Two-particle partition functions:* A two particle wave function is constructed from one particle states k_1, and k_2. Depending on the statistics of the two particles, we have

Bosons:

$$|k_1, k_2\rangle_B = \begin{cases} \left(|k_1\rangle|k_2\rangle + |k_2\rangle|k_1\rangle\right)/\sqrt{2} & \text{for } k_1 \neq k_2 \\ |k_1\rangle|k_2\rangle & \text{for } k_1 = k_2 \end{cases},$$

Fermions:

$$|k_1, k_2\rangle_F = \begin{cases} \left(|k_1\rangle|k_2\rangle - |k_2\rangle|k_1\rangle\right)/\sqrt{2} & \text{for } k_1 \neq k_2 \\ \text{no state} & \text{for } k_1 = k_2 \end{cases}.$$

For the Boson system, the partition function is given by

$$\begin{aligned}
Z_2^B &= \text{tr}\left(e^{-\beta H}\right) = \sum_{k_1,k_2} \langle k_1, k_2| e^{-\beta H} |k_1, k_2\rangle_B \\
&= \sum_{k_1>k_2} \frac{\langle k_1|\langle k_2| + \langle k_2|\langle k_1|}{\sqrt{2}} e^{-\beta H} \frac{|k_1\rangle|k_2\rangle + |k_2\rangle|k_1\rangle}{\sqrt{2}} + \sum_k \langle k|\langle k| e^{-\beta H} |k\rangle|k\rangle \\
&= \sum_{k_1>k_2} \exp\left[-\frac{\beta\hbar^2}{2m}(k_1^2 + k_2^2)\right] + \sum_k \exp\left(-\frac{2\beta\hbar^2 k^2}{2m}\right) \\
&= \frac{1}{2}\sum_{k_1,k_2} \exp\left[-\frac{\beta\hbar^2}{2m}(k_1^2 + k_2^2)\right] + \frac{1}{2}\sum_k \exp\left(-\frac{\beta\hbar^2 k^2}{m}\right),
\end{aligned}$$

and thus

$$Z_2^B = \frac{1}{2}\left[Z_1^2(m) + Z_1\left(\frac{m}{2}\right)\right].$$

For the Fermion system,

$$\begin{aligned}
Z_2^F &= \text{tr}\left(e^{-\beta H}\right) = \sum_{k_1,k_2} \langle k_1, k_2| e^{-\beta H} |k_1, k_2\rangle_F \\
&= \sum_{k_1>k_2} \frac{\langle k_1|\langle k_2| - \langle k_2|\langle k_1|}{\sqrt{2}} e^{-\beta H} \frac{|k_1\rangle|k_2\rangle - |k_2\rangle|k_1\rangle}{\sqrt{2}} \\
&= \frac{1}{2}\sum_{k_1,k_2} \exp\left[-\frac{\beta\hbar^2}{2m}(k_1^2 + k_2^2)\right] - \frac{1}{2}\sum_k \exp\left(-\frac{\beta\hbar^2 k^2}{m}\right),
\end{aligned}$$

and thus

$$Z_2^F = \frac{1}{2}\left[Z_1^2(m) - Z_1\left(\frac{m}{2}\right)\right].$$

Note that *classical Boltzmann particles* have a partition function

$$Z_2^{classical} = \frac{1}{2} Z_1(m)^2.$$

(b) If the system is non-degenerate, the correction term is much smaller than the classical term, and

$$\begin{aligned}
\ln Z_2^{\pm} &= \ln \left\{ \left[Z_1(m)^2 \pm Z_1(m/2) \right] /2 \right\} \\
&= 2 \ln Z_1(m) + \ln \left\{ 1 \pm \frac{Z_1(m/2)}{Z_1(m)^2} \right\} - \ln 2 \\
&\approx 2 \ln Z_1(m) \pm \frac{Z_1(m/2)}{Z_1(m)^2} - \ln 2.
\end{aligned}$$

Using

$$Z_1(m) = \frac{V}{\lambda(m)^3}, \quad \text{where} \quad \lambda(m) \equiv \frac{h}{\sqrt{2\pi m k_B T}},$$

we can write

$$\begin{aligned}
&\ln Z_2^{\pm} \approx \ln Z_2^{Classical} \pm \frac{\lambda(m)^3}{V} \left(\frac{\lambda(m)}{\lambda(m/2)} \right)^3 = \ln Z_2^{Classical} \pm 2^{-3/2} \frac{\lambda(m)^3}{V}, \\
\Longrightarrow \quad &\Delta \ln Z_2^{\pm} = \pm\, 2^{-3/2} \frac{h^3}{V(2\pi m k_B T)^{3/2}} = \pm\, 2^{-3/2} \frac{h^3 \beta^{3/2}}{V(2\pi m)^{3/2}}.
\end{aligned}$$

Thus the energy differences are

$$\Delta E^{\pm} = -\frac{\partial}{\partial \beta} \Delta \ln Z_2 = \mp \frac{3}{2^{5/2}} \frac{h^3 \beta^{3/2}}{V(2\pi m)^{3/2}} = \mp \frac{3}{2^{5/2}} k_B T \left(\frac{\lambda^3}{V} \right),$$

resulting in heat capacity differences of

$$\Delta C_V^{\pm} = \left. \frac{\partial \Delta E}{\partial T} \right|_V = \pm \frac{3}{2^{7/2}} \frac{h^3 k_B}{V(2\pi m k_B T)^{3/2}} = \pm \frac{3}{2^{7/2}} k_B \left(\frac{\lambda^3}{V} \right).$$

This approximation holds only when the thermal volume is much smaller than the volume of the gas where the ratio constitutes a small correction.

(c) *Quantum limit:* The approximation used in (b) is invalid if the correction terms are comparable to the first term. This occurs when the thermal wavelength becomes comparable to the size of the box, i.e. for

$$\lambda = \frac{h}{\sqrt{2\pi m k_B T}} \geq L \sim V^{1/3}, \quad \text{or} \quad T \leq \frac{h^2}{2\pi m k_B L^2}.$$

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2. *Solar interior:*

(a) *The thermal wavelengths* of electrons, protons, and α-particles in the sun are obtained from

$$\lambda = \frac{h}{\sqrt{2\pi m k_B T}},$$

and $T = 1.6 \times 10^{7\circ} K$, as

$$\begin{aligned}
\lambda_{\text{electron}} &\approx \frac{6.7 \times 10^{-34}\,\text{J/s}}{\sqrt{2\pi \times (9.1 \times 10^{-31}\,\text{Kg}) \cdot (1.4 \times 10^{-23}\,\text{J/K}) \cdot (1.6 \times 10^{7}\,\text{K})}} \approx 1.9 \times 10^{-11}\,\text{m},\\
\lambda_{\text{proton}} &\approx \frac{6.7 \times 10^{-34}\,\text{J/s}}{\sqrt{2\pi \times (1.7 \times 10^{-27}\,\text{Kg}) \cdot (1.4 \times 10^{-23}\,\text{J/K}) \cdot (1.6 \times 10^{7}\,\text{K})}} \approx 4.3 \times 10^{-13}\,\text{m},\\
\text{and} \qquad & \lambda_{\alpha-\text{particle}} = \frac{1}{2}\lambda_{\text{proton}} \approx 2.2 \times 10^{-13}\,\text{m}.
\end{aligned}$$

(b) *Degeneracy:* The corresponding number densities are given by

$$\begin{aligned}
\rho_H &\approx 6 \times 10^4\,\text{kg/m}^3 \quad \Longrightarrow \quad n_H \approx 3.5 \times 10^{31}\,\text{m}^{-3},\\
\rho_{He} &\approx 1.0 \times 10^5\,\text{Kg/m}^3 \quad \Longrightarrow \quad n_{He} = \frac{\rho_{He}}{4m_H} \approx 1.5 \times 10^{31}\,\text{m}^{-3},\\
n_e &= 2n_{He} + n_H \approx 8.5 \times 10^{31}\,\text{m}^3.
\end{aligned}$$

The criterion for degeneracy is $n\lambda^3 \geq 1$, and

$$\begin{aligned}
n_H \cdot \lambda_H^3 &\approx 2.8 \times 10^{-6} \ll 1,\\
n_{He} \cdot \lambda_{He}^3 &\approx 1.6 \times 10^{-7} \ll 1,\\
n_e \cdot \lambda_e^3 &\approx 0.58 \sim 1.
\end{aligned}$$

Thus the electrons are weakly degenerate, and the nuclei are not.

(c) *Pressure:* Since the nuclei are non-degenerate, and even the electrons are only weakly degenerate, their contributions to the overall pressure can be approximately calculated using the ideal gas law, as

$$\begin{aligned}
P &\approx (n_H + n_{he} + n_e) \cdot k_B T \approx 13.5 \times 10^{31}(\text{m}^{-3}) \cdot 1.38 \times 10^{-23}(\text{J/K}) \cdot 1.6 \times 10^{7}(\text{K})\\
&\approx 3.0 \times 10^{16}\,\text{N/m}^2.
\end{aligned}$$

(d) *The Radiation pressure* at the center of the sun can be calculated using the black body formulas,

$$P = \frac{1}{3}\frac{U}{V}, \qquad \text{and} \qquad \frac{1}{4}\frac{U}{V}c = \frac{\pi^2 k^4}{60\hbar^3 c^3} T^4 = \sigma T^4,$$

as

$$P = \frac{4}{3c}\sigma T^4 = \frac{4 \cdot 5.7 \times 10^{-8}\,\mathrm{W/(m^2K^4)} \cdot (1.6 \times 10^7\,\mathrm{K})^4}{3 \cdot 3.0 \times 10^8\,\mathrm{m/s}} \approx 1.7 \times 10^{13}\,\mathrm{N/m^2}.$$

Thus at the pressure in the solar interior is dominated by the particles.

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3. *Anharmonic dispersion*, with $\varepsilon = k^s$:

(a) *The grand partition function* is given by

Fermions:

$$Q_F = \sum_{N=0}^{\infty} e^{\beta\mu N} \sum_{\{n_\nu\}} e^{-\beta\sum_\nu n_\nu \varepsilon_\nu} = \prod_\nu \sum_{n_\nu=0}^{1} e^{\beta(\mu-\varepsilon_\nu)n_\nu} = \prod_\nu \left[1 + e^{\beta(\mu-\varepsilon_\nu)}\right].$$

Bosons:

$$Q_B = \sum_{N=0}^{\infty} e^{\beta\mu N} \sum_{\{n_\nu\}} e^{-\beta\sum_\nu n_\nu \varepsilon_\nu} = \prod_\nu \sum_{\{n_\nu\}} e^{\beta(\mu-\varepsilon_\nu)n_\nu} = \prod_\nu \left[1 - e^{\beta(\mu-\varepsilon_\nu)}\right]^{-1}.$$

Hence we obtain

$$\ln Q_\eta = -\eta \sum_\nu \ln\left[1 - \eta e^{\beta(\mu-\varepsilon_\nu)}\right],$$

where $\eta = +1$ for bosons, and $\eta = -1$ for fermions.

Changing the summation into an integration in d-dimensions yields

$$\sum_\nu \quad\rightarrow\quad \int d^d n \quad\rightarrow\quad \frac{V}{(2\pi)^d}\int d^d k = \frac{VS_d}{(2\pi)^d}\int k^{d-1}dk\,,$$

where

$$S_d = \frac{2\pi^{d/2}}{(d/2-1)!}.$$

Then

$$\begin{aligned}\ln Q_\eta &= -\eta\frac{VS_d}{(2\pi)^d}\int k^{d-1}dk\,\ln\left[1 - \eta e^{\beta(\mu-\varepsilon_\nu)}\right]\\ &= -\eta\frac{VS_d}{(2\pi)^d}\int k^{d-1}dk\,\ln\left[1 - \eta z e^{-\beta k^s}\right],\end{aligned}$$

where

$$z \equiv e^{\beta\mu}.$$

Introducing a new variable

$$x \equiv \beta k^s, \quad\rightarrow\quad k = \left(\frac{x}{\beta}\right)^{1/s}, \quad \text{and} \quad dk = \frac{1}{s}\left(\frac{x}{\beta}\right)^{1/s-1}\frac{dx}{\beta},$$

results in

$$\ln Q_\eta = -\eta \frac{VS_d}{(2\pi)^d} \frac{\beta^{-d/s}}{s} \int dx\, x^{d/s-1} \ln\left(1 - \eta z e^{-x}\right).$$

Integrating by parts,

$$\ln Q_\eta = \frac{VS_d}{(2\pi)^d} \frac{\beta^{-d/s}}{d} \int dx\, x^{d/s} \frac{z e^{-x}}{1 - \eta z e^{-x}} = \frac{VS_d}{(2\pi)^d} \frac{\beta^{-d/s}}{d} \int dx\, \frac{x^{d/s}}{z^{-1} e^{x} - \eta}.$$

Finally,

$$\ln Q_\eta = \frac{VS_d}{(2\pi)^d} \beta^{-d/s} \Gamma\left(\frac{d}{s} + 1\right) \cdot f^\eta_{d/s+1}(z),$$

where

$$f^\eta_n(z) \equiv \frac{1}{\Gamma(n)} \int dx\, \frac{x^{n-1}}{z^{-1} e^{x} - \eta}.$$

Using similar notation, the total particle number is obtained as

$$\begin{aligned}
N = \frac{\partial}{\partial(\beta\mu)} \ln Q_\eta \bigg|_\beta &= -\eta \frac{VS_d}{(2\pi)^d} \frac{\beta^{d/s}}{s} \frac{\partial}{\partial(\beta\mu)} \int dx\, x^{d/s-1} \ln\left(1 - \eta z e^{-x}\right) \\
&= \frac{VS_d}{(2\pi)^d} \frac{\beta^{d/s}}{s} \int dx\, \frac{x^{d/s-1} z e^{-x}}{1 - \eta z e^{-x}} = \frac{VS_d}{(2\pi)^d} \frac{\beta^{d/s}}{s} z \int dx\, \frac{x^{d/s-1}}{z^{-1} e^{-x} - \eta},
\end{aligned}$$

and thus

$$n = \frac{N}{V} = \frac{S_d}{(2\pi)^d s} (k_B T)^{d/s}\, \Gamma\left(\frac{d}{s}\right) \cdot f^\eta_{d/s}(z).$$

(b) *Equation of state:* The gas pressure is given by

$$PV = -k_B T \ln Q_\eta = \frac{VS_d}{(2\pi)^d} \frac{(k_B T)^{d/s+1}}{d}\, \Gamma\left(\frac{d}{s} + 1\right) \cdot f_{d/s+1}(z).$$

Note that since $\ln Q_\eta \propto \beta^{-d/s}$,

$$E = -\frac{\partial}{\partial \beta} \ln Q_\eta \bigg|_z = -\frac{d}{s} \frac{\ln Q_\eta}{\beta} = -\frac{d}{s} k_B T \ln Q_\eta, \quad \Longrightarrow \quad \frac{PV}{E} = \frac{s}{d},$$

which is the same result as in PS #7.

(c) For fermions at low T,

$$\frac{1}{e^{\beta(\varepsilon - \varepsilon_F)} + 1} \approx \theta(\varepsilon - \varepsilon_F), \quad \Longrightarrow \quad f_n(z) \approx \frac{(\beta \varepsilon_F)^n}{n \Gamma(n)},$$

resulting in

$$E \approx \frac{VS_d}{(2\pi)^d} \frac{\varepsilon_F^{d/s+1}}{d}, \quad PV \approx \frac{VS_d}{(2\pi)^d} \frac{\varepsilon_F^{d/s+1}}{s}, \quad n \approx \frac{S_d}{(2\pi)^d} \frac{\varepsilon_F^{d/s}}{s}.$$

Hence, we obtain

$$\frac{E}{N} \approx \frac{1}{n} \cdot \frac{S_d}{(2\pi)^d} \frac{\varepsilon_F^{d/s+1}}{d} \approx \frac{s}{d}\varepsilon_F \propto n^{s/d}, \quad \text{and}$$
$$P \propto \varepsilon_F^{d/s+1} \propto n^{s/d+1}.$$

(d) *Bose-Einstein condensation:* To see if Bose-Einstein condensation occurs, check whether a density n can be found such that $z = z_{max} = 1$. Since $f_{d/s}^{+}(z)$ is a monotonic function of z for $0 \le z \le 1$, the maximum value for the right hand side of the equation for $n = N/V$, given in (a) is

$$\frac{S_d}{(2\pi)^d s}(k_B T)^{d/s}\Gamma\left(\frac{d}{s}\right) f_{d/s}^{+}(1).$$

If this value is larger than n, we can always find $z < 1$ that satisfies the above equation. If this value is smaller than n, then the remaining portion of the particles should condense into the ground state. Thus the question is whether $f_{d/s}^{+}(1)$ diverges at $z = 1$, where

$$f_{d/s}^{+}(z_{max} = 1) = \frac{1}{\Gamma(d/s)} \int dx \, \frac{x^{d/s-1}}{e^x - 1}.$$

The integrand may diverge near $x = 0$; the contribution for $x \sim 0$ is

$$\int_0^{\varepsilon} dx \, \frac{x^{d/s-1}}{e^x - 1} \simeq \int_0^{\varepsilon} dx \, x^{d/s-2},$$

which converges for $d/s - 2 > -1$, or $d > s$. Therefore, Bose-Einstein condensation occurs for $d > s$. For a two dimensional gas, $d = s = 2$, the integral diverges logarithmically, and hence Bose-Einstein condensation does not occur.

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4. *Pauli's Paramagnetism* The energy of the electron gas is given by

$$E \equiv \sum_p E_p(n_p^{+}, n_p^{-}),$$

where $n_p^{\pm}$ (= 0 or 1), denote the number of particles having $\pm$ spins and momentum p, and

$$E_p(n_p^{+}, n_p^{-}) \equiv \left(\frac{p^2}{2m} - \mu_0 B\right) n_p^{+} + \left(\frac{p^2}{2m} + \mu_0 B\right) n_p^{-}$$
$$= (n_p^{+} + n_p^{-})\frac{p^2}{2m} - (n_p^{+} - n_p^{-})\mu_0 B.$$

(a) *The grand partition function* of the system is

$$\begin{aligned}
Q &= \sum_{N=0}^{\infty} \exp(-\beta\mu N) \sum_{\{n_p^+, n_p^-\}}^{N=\sum(n_p^+ + n_p^-)} \exp\left[-\beta E_p(n_p^+, n_p^-)\right] \\
&= \sum_{\{n_p^+, n_p^-\}} \exp\left[\beta\mu\left(n_p^+ + n_p^-\right) - \beta E_p\left(n_p^+, n_p^-\right)\right] \\
&= \prod_p \sum_{\{n_p^+, n_p^-\}} \exp\left\{\beta\left[\left(\mu - \mu_0 B - \frac{p^2}{2m}\right) n_p^+ + \left(\mu + \mu_0 B - \frac{p^2}{2m}\right) n_p^-\right]\right\} \\
&= \prod_p \left\{1 + \exp\left[\beta\left(\mu - \mu_0 B - \frac{p^2}{2m}\right)\right]\right\} \cdot \left\{1 + \exp\left[\beta\left(\mu + \mu_0 B - \frac{p^2}{2m}\right)\right]\right\} \\
&= Q_0\left(\mu + \mu_0 B\right) \cdot Q_0\left(\mu - \mu_0 B\right),
\end{aligned}$$

where

$$Q_0(\mu) \equiv \prod_p \left\{1 + \exp\left[\beta\left(\mu - \frac{p^2}{2m}\right)\right]\right\}.$$

Thus

$$\ln Q = \ln\left\{Q_0(\mu + \mu_0 B)\right\} + \ln\left\{Q_0(\mu - \mu_0 B)\right\}.$$

Each contribution is given by

$$\begin{aligned}
\ln Q_0(\mu) &= \sum_p \ln\left(1 + \exp\left[\beta(\mu - \frac{p^2}{2m})\right]\right) = \frac{V}{(2\pi\hbar)^3} \int d^3p \, \ln\left(1 + z e^{-\beta\frac{p^2}{2m}}\right) \\
&= \frac{V}{h^3} \frac{4\pi m}{\beta} \sqrt{\frac{2m}{\beta}} \int dx \, \sqrt{x} \ln(1 + z e^{-x}), \quad \text{where} \quad z \equiv e^{\beta\mu},
\end{aligned}$$

and integrating by parts yields

$$\ln Q_0(\mu) = V\left(\frac{2\pi m k_B T}{h^2}\right)^{3/2} \frac{2}{\sqrt{\pi}} \frac{2}{3} \int dx \, \frac{x^{3/2}}{z^{-1} e^x + 1} = \frac{V}{\lambda^3} f_{5/2}^-(z).$$

The total grand free energy is obtained from

$$\ln Q(\mu) = \frac{V}{\lambda^3}\left[f_{5/2}^-\left(z e^{\beta\mu_0 B}\right) + f_{5/2}^-\left(z e^{-\beta\mu_0 B}\right)\right],$$

as

$$G = -k_B T \ln Q(\mu) = -k_B T \frac{V}{\lambda^3}\left[f_{5/2}^-\left(z e^{\beta\mu_0 B}\right) + f_{5/2}^-\left(z e^{-\beta\mu_0 B}\right)\right].$$

(b) *The number densities* of electrons with up or down spins is given by

$$\frac{N_\pm}{V} = z \frac{\partial}{\partial z} \ln Q_\pm = \frac{V}{\lambda^3} f_{3/2}^-\left(z e^{\pm\beta\mu_0 B}\right),$$

where we used

$$z\frac{\partial}{\partial z}f_n^-(z) = f_{n-1}^-(z).$$

The total number of electrons is the sum of these, i.e.

$$N = N_+ + N_- = \frac{V}{\lambda^3}\left[f_{3/2}^-\big(ze^{\beta\mu_0 B}\big) + f_{3/2}^-\big(ze^{-\beta\mu_0 B}\big)\right]$$

(c) *The magnetization* is related to the difference between numbers of spin up and down electrons as

$$M = \mu_0(N_+ - N_-) = \mu_0\frac{V}{\lambda^3}\left[f_{3/2}^-\left(ze^{\beta\mu_0 B}\right) - f_{3/2}^-\left(ze^{-\beta\mu_0 B}\right)\right].$$

Expanding the results for small B, gives

$$f_{3/2}^-\left(ze^{\pm\beta\mu_0 B}\right) \approx f_{3/2}^-\left[z\left(1 \pm \beta\mu_0 B\right)\right] \approx f_{3/2}^-(z) \pm z\cdot\beta\mu_0 B\frac{\partial}{\partial z}f_{3/2}^-(z),$$

which results in

$$M = \mu_0\frac{V}{\lambda^3}(2\beta\mu_0 B)\cdot f_{1/2}^-(z) = \frac{2\mu_0^2}{k_B T}\frac{V}{\lambda^3}\cdot B\cdot f_{1/2}^-(z).$$

(d) *The magnetic susceptibility* is

$$\chi \equiv \left.\frac{\partial M}{\partial B}\right|_{B=0} = \frac{2\mu_0^2}{k_B T}\frac{V}{\lambda^3}\cdot f_{1/2}^-(z),$$

with z given by,

$$N = 2\frac{V}{\lambda^3}\cdot f_{3/2}^-(z).$$

In the low temperature limit, $(\ln z = \beta\mu \to \infty)$

$$f_n^-(z) = \frac{1}{\Gamma(n)}\int_0^\infty dx\,\frac{x^{n-1}}{1+e^{x-\ln(z)}} \quad \underset{T\to 0}{\approx} \quad \frac{1}{\Gamma(n)}\int_0^{\ln(z)} dx\,x^{n-1} = \frac{[\ln(z)]^n}{n\Gamma(n)},$$

$$N = 2\frac{V}{\lambda^3}\cdot\frac{4(\ln z)^{3/2}}{3\sqrt{\pi}}, \quad \Longrightarrow \quad \ln z = \left(\frac{3N\sqrt{\pi}}{8V}\lambda^3\right)^{2/3},$$

$$\chi = \frac{4\mu_0^2 V}{\sqrt{\pi}k_B T\lambda^3}\cdot\left(\frac{3N\sqrt{\pi}}{8V}\lambda^3\right)^{1/3} = \frac{2\mu_o^2 V}{k_B T\lambda^2}\cdot\left(\frac{3N}{\pi V}\right)^{1/3} = \frac{4\pi m\mu_0^2 V}{h^2}\cdot\left(\frac{3N}{\pi V}\right)^{1/3}.$$

Their ratio of the last two expressions gives

$$\left.\frac{\chi}{N}\right|_{T\to 0} = \frac{\mu_0^2}{k_B T}\frac{f_{1/2}^-}{f_{3/2}^-} = \frac{3\mu_0^2}{2k_B T}\frac{1}{\ln(z)} = \frac{3\mu_0^2}{2k_B T}\frac{1}{\beta\varepsilon_F} = \frac{3\mu_0^2}{2k_B T_F}.$$

In the high temperature limit ($z \to 0$),

$$f_n(z) \underset{z\to 0}{\longrightarrow} \frac{z}{\Gamma(n)} \int_0^\infty dx\, x^{n-1} e^{-x} = z,$$

and thus

$$N \underset{\beta\to 0}{\longrightarrow} \frac{2V}{\lambda^3} \cdot z, \quad \Longrightarrow \quad z \approx \frac{N}{2V} \cdot \lambda^3 = \frac{N}{2V} \cdot \left(\frac{h^2}{2\pi m k_B T}\right)^{3/2} \to 0,$$

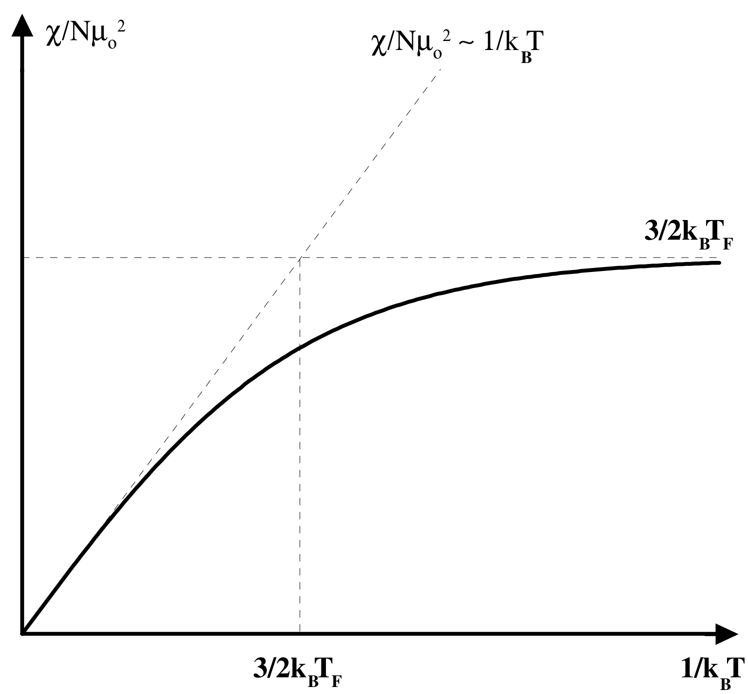
which is consistent with $\beta \to 0$. Using this result,

$$\chi \approx \frac{2\mu_0^2 V}{k_B T \lambda^3} \cdot z = \frac{N\mu_0^2}{k_B T}.$$

The result

$$\left(\frac{\chi}{N}\right)_{T\to\infty} = \frac{\mu_0^2}{k_B T},$$

is known as the *Curie susceptibility.*



(e) χ/N *for a typical metal-* Since $T_{Room} \ll T_F \approx 10^{4\circ} K$, we can take the low T limit for χ (see(d)), and

$$\frac{\chi}{N} = \frac{3\mu_0^2}{2k_B T_F} \approx \frac{3 \times (9.3 \times 10^{-24})^2}{2 \times 1.38 \times 10^{-23}} \approx 9.4 \times 10^{-24} \mathrm{J/T}^2.$$

where we used

$$\mu_0 = \frac{eh}{2mc} \simeq 9.3 \times 10^{-24}\,\mathrm{J/T}.$$

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5. *Fermi Energy, Fermi Temperature, & Heat Capacities*

(a) *Fermi temperature:* Using $g = 2$ for spin 1/2 particles, and

$$\varepsilon_F = \frac{\hbar^2}{2m}\left(3\pi^2 n\right)^{2/3}, \quad \Longrightarrow \quad T_F \equiv \frac{\varepsilon_F}{k_B},$$

we can construct the following table:

	$n(1/\mathrm{m}^3)$	$m(\mathrm{Kg})$	$\varepsilon_F(\mathrm{eV})$	$T_F(^\circ K)$
electron	10^{29}	9×10^{-31}	4.4	5×10^4
nucleons	10^{44}	1.6×10^{-27}	1.0×10^8	1.1×10^{12}
liquid He^3	2.6×10^{28}	4.6×10^{-27}	$\times 10^{-3}$	10^1

(b) *Heat capacity of phonon (solid) and electron gases:* For an electron gas, $T_F \approx 5{\times}10^{4\circ}K$,

$$T_F \gg T_{\text{room}}, \quad \Longrightarrow \quad \frac{C_{\text{electron}}}{Nk_B} \approx \frac{\pi^2}{2} \cdot \frac{T}{T_F} \approx 0.025.$$

For the phonon gas in iron, the Debye temperature is $T_D \approx 470^\circ K$, and hence

$$\frac{C_{\text{phonon}}}{Nk_B} \approx 3\left[1 - \frac{1}{20}\left(\frac{T}{T_D}\right)^2 + \dots\right] \approx 3,$$

resulting in

$$\frac{C_{\text{electron}}}{C_{\text{phonon}}} \approx 8 \times 10^{-3}.$$

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