

1. Numerical estimates

(a) *Thermal wavelengths* are given by

$$\lambda \equiv \frac{h}{\sqrt{2\pi m k_B T}}.$$

For a neutron at room temperature, using the values

$$m = 1.67 \times 10^{-27} \text{kg}, \quad T = 300^\circ \text{K},$$

$$k_B = 1.38 \times 10^{-23} \text{JK}^{-1}, \quad h = 6.67 \times 10^{-34} \text{Js},$$

we obtain $\lambda = 1 \text{\AA}$.

The typical wavelength of a phonon in a solid is $\lambda = 0.01 m$, which is much longer than the neutron wavelength. The minimum wavelength is, however, of the order of atomic spacing ($3 - 5 \text{\AA}$), which is comparable to the neutron thermal wavelength.

(b) *Degeneracy criterion:* Quantum mechanical effects become important if $n\lambda^3 \geq 1$. In the high temperature limit the ideal gas law is valid, and the degeneracy criterion can be reexpressed in terms of pressure $P = nk_B T$, as

$$n\lambda^3 = \frac{nh^3}{(2\pi m k_B T)^{3/2}} = \frac{P}{(k_B T)^{5/2}} \frac{h^3}{(2\pi m)^{3/2}} \ll 1.$$

It is convenient to express the answers starting with an imaginary gas of ‘protons’ at room temperature and pressure, for which

$$m_p = 1.7 \times 10^{-34} \text{Kg}, \quad P = 1 \text{ atm.} = 10^5 \text{ Nm}^{-2},$$

$$\text{and } (n\lambda^3)_{\text{proton}} = \frac{10^{-5}}{(4.1 \times 10^{-21})^{5/2}} \frac{(6.7 \times 10^{-34})^3}{(2\pi \cdot 1.7 \times 10^{-27})^{3/2}} = 2 \times 10^{-5}.$$

The quantum effects appear below $T = T_Q$, at which $n\lambda^3$ becomes order of unity. Using

$$n\lambda^3 = (n\lambda^3)_{\text{proton}} \left(\frac{m_p}{m} \right)^{3/2}, \quad \text{and} \quad T_Q = T_{\text{room}} (n\lambda^3)^{3/2},$$

we obtain the following table:

		m/m_p	$n\lambda^3$	$T_Q(K)$
Hydrogen	H_2	2	0.7×10^{-5}	2.6
Helium	He	4	3.0×10^{-6}	1.9
Oxygen	O_2	32	0.1×10^{-6}	0.5

(c) *Dispersion relation:* A spectrum of low energy excitations scaling as

$$\mathcal{E}(k) \propto k^s,$$

in d -dimensional space, leads to a low temperature heat capacity that vanishes as

$$C \propto T^{d/s}.$$

Therefore, from $C_V = 20.4 T^3 \text{JKg}^{-1} \text{ } ^\circ K^{-1}$ in $d = 3$, we can conclude $s = 1$, i.e. a spectrum of the form

$$\mathcal{E}(k) = \hbar c_s |\vec{k}|,$$

corresponding to sound waves of speed c_s . Inserting all the numerical factors, we have

$$C_V = \frac{12\pi^4 N k_B}{5} \left(\frac{T}{\Theta} \right)^3, \quad \text{where} \quad \Theta = \frac{\hbar c_s}{k_B} \left(\frac{6\pi^2 N}{V} \right)^{1/3}.$$

Hence, we obtain

$$\mathcal{E} = \hbar c_s k = k_B \left(\frac{2\pi^2 k_B V}{5} \frac{T^3}{C_V} \right)^{1/3} k = (2 \times 10^{-32} \text{Jm}) k,$$

corresponding to a sound speed of $c_s \approx 2 \times 10^2 \text{ms}^{-1}$.

2. Solid-liquid transition of He^3

(a) *Entropy of solid He^3* comes from the nuclear spin degeneracies, and is given by

$$s_s = \frac{S_s}{N} = \frac{k_B \ln(2^N)}{N} = k_B \ln 2.$$

(b) *Fermi temperature* for liquid ^3He may be obtained from its density as

$$\begin{aligned} T_F &= \frac{\varepsilon_F}{k_B} = \frac{\hbar^2}{2mk_B} \left(\frac{3N}{8\pi V} \right)^{2/3} \\ &\approx \frac{(6.7 \times 10^{-34})^2}{2 \cdot (6.8 \times 10^{-27})(1.38 \times 10^{-23})} \left(\frac{3}{8\pi \times 46 \times 10^{-30}} \right)^{2/3} \approx 9.2 \text{ } ^\circ K. \end{aligned}$$

(c) *Heat capacity* comes from the density of excited states at the fermi surface, and is given by

$$C_V = k_B \frac{\pi^2}{6} k_B T D(\varepsilon_F) = \frac{\pi^2}{6} k_B^2 T \frac{3N}{2k_B T_F} = \frac{\pi^2}{4} N k_B \frac{T}{T_F}.$$

(d) *Entropy* can be obtained from the heat capacity as

$$C_V = \frac{TdS}{dT}, \quad \Rightarrow \quad s_\ell = \frac{1}{N} \int_0^T \frac{C_V dT}{T} = \frac{\pi^2}{4} k_B \frac{T}{T_F}.$$

As $T \rightarrow 0$, $s_\ell \rightarrow 0$, while s_s remains finite. This is an unusual situation in which the solid has more entropy than the liquid! (The finite entropy is due to treating the nuclear spins as independent. There is actually a weak coupling between spins which causes magnetic ordering at a much lower temperature, removing the finite entropy.)

(e) *Clausius-Clapeyron equation* can be obtained by equating the chemical potentials at the phase boundary,

$$\mu_\ell(T, P) = \mu_s(T, P), \quad \text{and} \quad \mu_\ell(T + \Delta T, P + \Delta P) = \mu_s(T + \Delta T, P + \Delta P).$$

Expanding the second equation, and using the thermodynamic identities

$$\left(\frac{\partial \mu}{\partial T} \right)_P = S, \quad \text{and} \quad \left(\frac{\partial \mu}{\partial P} \right)_T = -V,$$

results in

$$\left(\frac{\partial P}{\partial T} \right)_{\text{melting}} = \frac{s_\ell - s_s}{v_\ell - v_s}.$$

(f) *The negative slope* of the phase boundary results from the solid having more entropy than the liquid, and can be calculated from the Clausius-Clapeyron relation

$$\left(\frac{\partial P}{\partial T} \right)_{\text{melting}} = \frac{s_\ell - s_s}{v_\ell - v_s} \approx k_B \frac{\frac{\pi^2}{4} \left(\frac{T}{T_F} \right) - \ln 2}{v_\ell - v_s}.$$

Using the values, $T = 0.1 \text{ }^\circ\text{K}$, $T_F = 9.2 \text{ J }^\circ\text{K}$, and $v_\ell - v_s = 3 \text{ \AA}^3$, we estimate

$$\left(\frac{\partial P}{\partial T} \right)_{\text{melting}} \approx -2.7 \times 10^6 \text{ Pa }^\circ\text{K}^{-1},$$

in reasonable agreement with the observations.

3. Boson Ferromagnetism:

(a) *Average occupation numbers* of the one-particle states in the grand canonical ensemble of chemical potential μ , are given by the Bose-Einstein distribution

$$\begin{aligned} n_s(\vec{k}) &= \frac{1}{e^{\beta[\mathcal{H}(s)-\mu]} - 1}, \quad (\text{for } s = -1, 0, 1) \\ &= \frac{1}{\exp\left[\beta\left(\frac{\hbar^2 k^2}{2m} - \mu_0 s B\right) - \beta\mu\right] - 1}. \end{aligned}$$

(b) *Total numbers of particles with spin s* are given by

$$N_s = \sum_{\{\vec{k}\}} n_s(\vec{k}), \quad \Rightarrow \quad N_s = \frac{V}{(2\pi)^3} \int d^3k \frac{1}{\exp\left[\beta\left(\frac{\hbar^2 k^2}{2m} - \mu_0 s B\right) - \beta\mu\right] - 1}.$$

After a change of variables, $k \equiv x^{1/2} \sqrt{2mk_B T}/h$, we get

$$N_s = \frac{V}{\lambda^3} f_{3/2}^+(ze^{\beta\mu_0 s B}),$$

where

$$f_m^+(z) \equiv \frac{1}{\Gamma(m)} \int_0^\infty \frac{dx x^{m-1}}{z^{-1}e^x - 1}, \quad \lambda \equiv \frac{h}{\sqrt{2\pi m k_B T}}, \quad z \equiv e^{\beta\mu}.$$

(c) *Magnetization* is obtained from

$$\begin{aligned} M(T, \mu) &= \mu_0 (N_+ - N_-) \\ &= \mu_0 \frac{V}{\lambda^3} \left[f_{3/2}^+(ze^{\beta\mu_0 B}) - f_{3/2}^+(ze^{-\beta\mu_0 B}) \right]. \end{aligned}$$

Expanding the result for small B gives

$$f_{3/2}^+(ze^{\pm\beta\mu_0 B}) \approx f_{3/2}^+(z[1 \pm \beta\mu_0 B]) \approx f_{3/2}^+(z) \pm z \cdot \beta\mu_0 B \frac{\partial}{\partial z} f_{3/2}^+(z).$$

Using $z df_m^+(z)/dz = f_{m-1}^+(z)$, we obtain

$$M = \mu_0 \frac{V}{\lambda^3} (2\beta\mu_0 B) \cdot f_{1/2}^+(z) = \frac{2\mu_0^2}{k_B T} \frac{V}{\lambda^3} \cdot B \cdot f_{1/2}^+(z),$$

and

$$\chi \equiv \left. \frac{\partial M}{\partial B} \right|_{B=0} = \frac{2\mu_0^2}{k_B T} \frac{V}{\lambda^3} \cdot f_{1/2}^+(z).$$

(d) *In the high temperature limit*, z is small. Use the Taylor expansion for $f_m^+(z)$ to write the total density $n(B=0)$, as

$$\begin{aligned} n(B=0) &= \frac{N_+ + N_0 + N_-}{V} \Big|_{B=0} = \frac{3}{\lambda^3} f_{3/2}^+(z) \\ &\approx \frac{3}{\lambda^3} \left(z + \frac{z^2}{2^{3/2}} + \frac{z^3}{3^{3/2}} + \dots \right). \end{aligned}$$

Inverting the above equation gives

$$z = \left(\frac{n\lambda^3}{3} \right) - \frac{1}{2^{3/2}} \left(\frac{n\lambda^3}{3} \right)^2 + \dots$$

The susceptibility is then calculated as

$$\begin{aligned} \chi &= \frac{2\mu_0^2}{k_B T} \frac{V}{\lambda^3} \cdot f_{1/2}^+(z), \\ \chi/N &= \frac{2\mu_0^2}{k_B T} \frac{1}{n\lambda^3} \left(z + \frac{z^2}{2^{1/2}} + \dots \right) \\ &= \frac{2\mu_0^2}{3k_B T} \left[1 + \left(-\frac{1}{2^{3/2}} + \frac{1}{2^{1/2}} \right) \left(\frac{n\lambda^3}{3} \right) + O(n^2) \right]. \end{aligned}$$

(e) *Bose-Einstein condensation* occurs when $z=1$, at a density

$$n = \frac{3}{\lambda^3} f_{3/2}^+(1),$$

or a temperature

$$T_c(n) = \frac{h^2}{2\pi m k_B} \left(\frac{n}{3\zeta_{3/2}} \right)^{2/3},$$

where $\zeta_{3/2} \equiv f_{3/2}^+(1) \approx 2.61$.

Since $\lim_{z \rightarrow 1} f_{1/2}^+(z) = \infty$, the susceptibility $\chi(T, n)$ diverges on approaching $T_c(n)$ from the high temperature side.

(f) *Chemical potential for $T < T_c$* : Since $n_s(\vec{k}, B) = [z^{-1} e^{\beta \mathcal{E}_s(\vec{k}, B)} - 1]^{-1}$ is a positive number for all \vec{k} and s_z , μ is bounded above by the minimum possible energy, i.e.

$$\text{for } T < T_c, \text{ and } B \text{ finite, } z e^{\beta \mu_0 B} = 1, \implies \mu = -\mu_0 B.$$

Hence the macroscopically occupied one particle state has $\vec{k} = 0$, and $s_z = +1$.

(g) *Spontaneous magnetization*: Contribution of the excited states to the magnetization vanishes as $B \rightarrow 0$. Therefore the total magnetization for $T < T_c$ is due to the macroscopic occupation of the $(k=0, s_z = +1)$ state, and

$$\begin{aligned} \overline{M}(T, n) &= \mu_0 V n_+(k=0) \\ &= \mu_0 V (n - n_{excited}) = \mu_0 \left(N - \frac{3V}{\lambda^3} \zeta_{3/2} \right). \end{aligned}$$
