Two–dimensional electron gas

1. *Electron gas in a magnetic field:*

(a) The Hamiltonian for non–interacting free electrons in a magnetic field has the form

$$\mathcal{H} = \sum_i \left[\frac{\left(\vec{p}_i + e\vec{A}\right)^2}{2m} \pm \mu_B |\vec{B}| \right],$$

or in expanded form

$$\mathcal{H} = \frac{p^2}{2m} + \frac{e}{m}\vec{p}\cdot\vec{A} + \frac{e^2}{2m}\vec{A}^{\,2} \pm \mu_B|\vec{B}|.$$

Substituting $\vec{A} = \vec{B}\times\vec{q}/2$, results in

$$\begin{aligned}\mathcal{H} &= \frac{p^2}{2m} + \frac{e}{2m}\vec{p}\cdot\vec{B}\times\vec{q} + \frac{e^2}{8m}\left(\vec{B}\times\vec{q}\right)^2 \pm \mu_B|\vec{B}| \\ &= \frac{p^2}{2m} + \frac{e}{2m}\vec{p}\times\vec{B}\cdot\vec{q} + \frac{e^2}{8m}\left(B^2q^2 - (\vec{B}\cdot\vec{q})^2\right) \pm \mu_B|\vec{B}|.\end{aligned}$$

Using the canonical equations, $\dot{\vec{q}} = \partial\mathcal{H}/\vec{p}$ and $\dot{\vec{p}} = -\partial\mathcal{H}/\vec{q}$, we find

$$\begin{cases} \dot{\vec{q}} = \dfrac{\partial\mathcal{H}}{\partial\vec{p}} = \dfrac{\vec{p}}{m} + \dfrac{e}{2m}\vec{B}\times\vec{q}, \quad \Longrightarrow \quad \vec{p} = m\dot{\vec{q}} - \dfrac{e}{2}\vec{B}\times\vec{q}, \\ \dot{\vec{p}} = -\dfrac{\partial\mathcal{H}}{\partial\vec{q}} = -\dfrac{e}{2m}\vec{p}\times\vec{B} - \dfrac{e^2}{4m}B^2\vec{q} + \dfrac{e^2}{4m}\left(\vec{B}\cdot\vec{q}\right)\vec{B}. \end{cases}$$

Differentiating the first expression obtained for $\vec{p}$, and setting it equal to the second expression for $\dot{\vec{p}}$ above, gives

$$m\ddot{\vec{q}} - \frac{e}{2}\vec{B}\times\dot{\vec{q}} = -\frac{e}{2m}\left(m\dot{\vec{q}} - \frac{e}{2}\vec{B}\times\vec{q}\right)\times\vec{B} - \frac{e^2}{4m}|\vec{B}|^2\vec{q} + \frac{e^2}{4m}\left(\vec{B}\cdot\vec{q}\right)\vec{B}.$$

Simplifying the above expression, using $\vec{B}\times\vec{q}\times\vec{B} = B^2\vec{q} - \left(\vec{B}\cdot\vec{q}\right)\vec{B}$, leads to

$$m\ddot{\vec{q}} = e\vec{B}\times\dot{\vec{q}}.$$

This describes the rotation of electrons in cyclotron orbits,

$$\ddot{\vec{q}} = \vec{\omega}_c\times\dot{\vec{q}},$$

where $\vec{\omega}_c = e\vec{B}/m$; i.e. rotations are in the plane perpendicular to $\vec{B}$.

(b) Consider the classes of collisions described by cross-sections $\sigma \equiv \sigma_{\uparrow\uparrow} = \sigma_{\downarrow\downarrow}$, and $\sigma_\times \equiv \sigma_{\uparrow\downarrow}$. We can write the Boltzmann equations for the densities as

$$\frac{\partial f_\uparrow}{\partial t} + \{\mathcal{H}_\uparrow, f_\uparrow\} = \int d^2p_2 d\Omega |v_1 - v_2| \left\{ \frac{d\sigma}{d\Omega} \left[f_\uparrow(\vec{p_1}\,') f_\uparrow(\vec{p_2}\,') - f_\uparrow(\vec{p_1}) f_\uparrow(\vec{p_2}) \right] + \right.$$
$$\left. \frac{d\sigma_\times}{d\Omega} \left[f_\uparrow(\vec{p_1}\,') f_\downarrow(\vec{p_2}\,') - f_\uparrow(\vec{p_1}) f_\downarrow(\vec{p_2}) \right] \right\},$$

and

$$\frac{\partial f_\downarrow}{\partial t} + \{\mathcal{H}_\downarrow, f_\downarrow\} = \int d^2p_2 d\Omega |v_1 - v_2| \left\{ \frac{d\sigma}{d\Omega} \left[f_\downarrow(\vec{p_1}\,') f_\downarrow(\vec{p_2}\,') - f_\downarrow(\vec{p_1}) f_\downarrow(\vec{p_2}) \right] + \right.$$
$$\left. \frac{d\sigma_\times}{d\Omega} \left[f_\downarrow(\vec{p_1}\,') f_\uparrow(\vec{p_2}\,') - f_\downarrow(\vec{p_1}) f_\uparrow(\vec{p_2}) \right] \right\}.$$

(c) The usual Boltzmann H–Theorem states that $d\mathrm{H}/dt \le 0$, where

$$\mathrm{H} = \int d^2q d^2p f(\vec{q}, \vec{p}, t) \ln f(\vec{q}, \vec{p}, t).$$

For the electron gas in a magnetic field, the H function can be generalized to

$$\mathrm{H} = \int d^2q d^2p \left[f_\uparrow \ln f_\uparrow + f_\downarrow \ln f_\downarrow \right],$$

where the condition $d\mathrm{H}/dt \le 0$ is proved as follows:

$$\begin{aligned} \frac{d\mathrm{H}}{dt} &= \int d^2q d^2p \left[\frac{\partial f_\uparrow}{\partial t} (\ln f_\uparrow + 1) + \frac{\partial f_\downarrow}{\partial t} (\ln f_\downarrow + 1) \right] \\ &= \int d^2q d^2p \left[(\ln f_\uparrow + 1) \left(\{f_\uparrow, \mathcal{H}_\uparrow\} + C_{\uparrow\uparrow} + C_{\uparrow\downarrow} \right) + (\ln f_\downarrow + 1) \left(\{f_\downarrow, \mathcal{H}_\downarrow\} + C_{\downarrow\downarrow} + C_{\downarrow\uparrow} \right) \right], \end{aligned}$$

with $C_{\uparrow\uparrow}$, etc., defined via the right hand side of the equations in part (b). Hence

$$\begin{aligned} \frac{d\mathrm{H}}{dt} &= \int d^2q d^2p \, (\ln f_\uparrow + 1) (C_{\uparrow\uparrow} + C_{\uparrow\downarrow}) + (\ln f_\downarrow + 1) (C_{\downarrow\downarrow} + C_{\downarrow\uparrow}) \\ &= \int d^2q d^2p \, (\ln f_\uparrow + 1) C_{\uparrow\uparrow} + (\ln f_\downarrow + 1) C_{\downarrow\downarrow} + (\ln f_\uparrow + 1) C_{\uparrow\downarrow} + (\ln f_\downarrow + 1) C_{\downarrow\uparrow} \\ &\equiv \frac{d\mathrm{H}_{\uparrow\uparrow}}{dt} + \frac{d\mathrm{H}_{\downarrow\downarrow}}{dt} + \frac{d}{dt} \left(\mathrm{H}_{\uparrow\downarrow} + \mathrm{H}_{\downarrow\uparrow} \right), \end{aligned}$$

where the $\mathrm{H}'s$ are in correspondence to the integrals for the collisions. We have also made use of the fact that $\int d^2p d^2q \{f_\uparrow, \mathcal{H}_\uparrow\} = \int d^2p d^2q \{f_\downarrow, \mathcal{H}_\downarrow\} = 0$. Dealing with each of the terms in the final equation individually,

$$\frac{d\mathrm{H}_{\uparrow\uparrow}}{dt} = \int d^2q d^2p_1 d^2p_2 d\Omega |v_1 - v_2| (\ln f_\uparrow + 1) \frac{d\sigma}{d\Omega} \left[f_\uparrow(\vec{p_1}\,') f_\uparrow(\vec{p_2}\,') - f_\uparrow(\vec{p_1}) f_\uparrow(\vec{p_2}) \right].$$

After symmetrizing this equation, as done in lectures and notes,

$$\frac{d\mathrm{H}_{\uparrow\uparrow}}{dt} = -\frac{1}{4}\int d^2q d^2p_1 d^2p_2 d\Omega |v_1 - v_2| \frac{d\sigma}{d\Omega} \left[\ln f_\uparrow(\vec{p}_1) f_\uparrow(\vec{p}_2) - \ln f_\uparrow(\vec{p}_1\ ') f_\uparrow(\vec{p}_2\ ')\right]$$
$$\cdot \left[f_\uparrow(\vec{p}_1) f_\uparrow(\vec{p}_2) - f_\uparrow(\vec{p}_1\ ') f_\uparrow(\vec{p}_2\ ')\right] \leq 0.$$

Similarly, $d\mathrm{H}_{\downarrow\downarrow}/dt \leq 0$. Dealing with the two remaining terms,

$$\begin{aligned}\frac{d\mathrm{H}_{\uparrow\downarrow}}{dt} &= \int d^2q d^2p_1 d^2p_2 d\Omega |v_1 - v_2| \left[\ln f_\uparrow(\vec{p}_1) + 1\right] \frac{d\sigma_\times}{d\Omega} \left[f_\uparrow(\vec{p}_1\ ') f_\downarrow(\vec{p}_2\ ') - f_\uparrow(\vec{p}_1) f_\downarrow(\vec{p}_2)\right] \\ &= \int d^2q d^2p_1 d^2p_2 d\Omega |v_1 - v_2| \left[\ln f_\uparrow(\vec{p}_1\ ') + 1\right] \frac{d\sigma_\times}{d\Omega} \left[f_\uparrow(\vec{p}_1) f_\downarrow(\vec{p}_2) - f_\uparrow(\vec{p}_1\ ') f_\downarrow(\vec{p}_2\ ')\right],\end{aligned}$$

where we have exchanged $(\vec{p}_1, \vec{p}_2 \leftrightarrow \vec{p}_1\ ', \vec{p}_2\ ')$. Averaging these two expressions together,

$$\frac{d\mathrm{H}_{\uparrow\downarrow}}{dt} = -\frac{1}{2}\int d^2q d^2p_1 d^2p_2 d\Omega |v_1 - v_2| \frac{d\sigma_\times}{d\Omega} \left[\ln f_\uparrow(\vec{p}_1) - \ln f_\uparrow(\vec{p}_1\ ')\right]$$
$$\cdot \left[f_\uparrow(\vec{p}_1) f_\downarrow(\vec{p}_2) - f_\uparrow(\vec{p}_1\ ') f_\downarrow(\vec{p}_2\ ')\right].$$

Likewise

$$\frac{d\mathrm{H}_{\downarrow\uparrow}}{dt} = -\frac{1}{2}\int d^2q d^2p_1 d^2p_2 d\Omega |v_1 - v_2| \frac{d\sigma_\times}{d\Omega} \left[\ln f_\downarrow(\vec{p}_2) - \ln f_\downarrow(\vec{p}_2\ ')\right]$$
$$\cdot \left[f_\downarrow(\vec{p}_2) f_\uparrow(\vec{p}_1) - f_\downarrow(\vec{p}_2\ ') f_\uparrow(\vec{p}_1\ ')\right].$$

Combining these two expressions,

$$\frac{d}{dt}\left(\mathrm{H}_{\uparrow\downarrow} + \mathrm{H}_{\downarrow\uparrow}\right) = -\frac{1}{4}\int d^2q d^2p_1 d^2p_2 d\Omega |v_1 - v_2| \frac{d\sigma_\times}{d\Omega}$$
$$\left[\ln f_\uparrow(\vec{p}_1) f_\downarrow(\vec{p}_2) - \ln f_\uparrow(\vec{p}_1\ ') f_\downarrow(\vec{p}_2\ ')\right] \left[f_\uparrow(\vec{p}_1) f_\downarrow(\vec{p}_2) - f_\uparrow(\vec{p}_1\ ') f_\downarrow(\vec{p}_2\ ')\right] \leq 0.$$

Since each contribution is separately negative, we have

$$\frac{d\mathrm{H}}{dt} = \frac{d\mathrm{H}_{\uparrow\uparrow}}{dt} + \frac{d\mathrm{H}_{\downarrow\downarrow}}{dt} + \frac{d}{dt}\left(\mathrm{H}_{\uparrow\downarrow} + \mathrm{H}_{\downarrow\uparrow}\right) \leq 0.$$

(d) For $d\mathrm{H}/dt = 0$ we need each of the three square brackets in the previous derivation to be zero. The first two contributions, from $d\mathrm{H}_{\uparrow\downarrow}/dt$ and $d\mathrm{H}_{\downarrow\downarrow}/dt$, are similar to those discussed in the notes for a single particle, and vanish for any $\ln f$ which is a linear combination of quantities conserved in collisions

$$\ln f_\alpha = \sum_i a_i^\alpha(\vec{q}) \chi_i(\vec{p}),$$

where $\alpha = (\uparrow \quad \text{or} \quad \downarrow)$. Clearly at each location $\vec{q}$, for such f_α,

$$\ln f_\alpha(\vec{p}_1) + \ln f_\alpha(\vec{p}_2) = \ln f_\alpha(\vec{p}_1{}') + \ln f_\alpha(\vec{p}_2{}').$$

If we consider only the first two terms of $d\mathrm{H}/dt = 0$, the coefficients $a_i^\alpha(\vec{q})$ can vary with both $\vec{q}$ and $\alpha = (\uparrow \quad \text{or} \quad \downarrow)$. This changes when we consider the third term $d\left(\mathrm{H}_{\uparrow\downarrow} + \mathrm{H}_{\downarrow\uparrow}\right)/dt$. The conservations of momentum and kinetic energy constrain the corresponding four functions to be the same, i.e. they require $a_i^\uparrow(\vec{q}) = a_i^\downarrow(\vec{q})$. There is, however, no similar constraint for the overall constant that comes from particle number conservation, as the numbers of spin-up and spin-down particles is *separately* conserved, i.e. $a_0^\uparrow(\vec{q}) = a_0^\downarrow(\vec{q})$. This implies that the densities of up and down spins can be different in the final equilibrium, while the two systems must share the same velocity and temperature.

(e) The Boltzmann equation is

$$\frac{\partial f_\alpha}{\partial t} = \{f_\alpha, \mathcal{H}_\alpha\} + C_{\alpha\alpha} + C_{\alpha\beta},$$

where the right hand side consists of streaming terms $\{f_\alpha, \mathcal{H}_\alpha\}$, and collision terms C. Let I_i denote any quantity conserved by the one body Hamiltonian, i.e. $\{I_i\,, \mathcal{H}_\alpha\} = 0$. Consider f_α which is a function only of the $I_i's$

$$f_\alpha \equiv f_\alpha\left(I_1, I_2, \cdots\right).$$

Then

$$\{f_\alpha, \mathcal{H}_\alpha\} = \sum_j \frac{\partial f_\alpha}{\partial I_j}\{I_j\,, \mathcal{H}_\alpha\} = 0.$$

(f) Conservation of momentum for a collision at $\vec{q}$

$$(\vec{p}_1 + \vec{p}_2) = (\vec{p}_1{}' + \vec{p}_2{}'),$$

implies

$$\vec{q} \times (\vec{p}_1 + \vec{p}_2) = \vec{q} \times (\vec{p}_1{}' + \vec{p}_2{}'),$$

or

$$\vec{L}_1 + \vec{L}_2 = \vec{L}_1{}' + \vec{L}_2{}',$$

where we have used $\vec{L}_i = \vec{q} \times \vec{p}_i$. Hence angular momentum $\vec{L}$ is conserved during collisions. Note that only the z–component L_z is present for electrons moving in 2–dimensions,

$\vec{q} \equiv (x_1, x_2)$, as is the case for the electron gas studied in this problem. Consider the Hamiltonian discussed in (a)

$$\mathcal{H} = \frac{p^2}{2m} + \frac{e}{2m}\vec{p} \times \vec{B} \cdot \vec{q} + \frac{e^2}{8m}\left(B^2 q^2 - (\vec{B} \cdot \vec{q})^2\right) \pm \mu_B |\vec{B}|.$$

Let us evaluate the Poisson brackets of the individual terms with $L_z = \vec{q} \times \vec{p} \;\big|_z$. The first term is

$$\left\{|\vec{p}|^2, \vec{q} \times \vec{p}\right\} = \varepsilon_{ijk}\left\{p_l p_l, x_j p_k\right\} = \varepsilon_{ijk} 2 p_l \frac{\partial}{\partial x_l}(x_j p_k) = 2\varepsilon_{ilk} p_l p_k = 0,$$

where we have used $\varepsilon_{ijk} p_j p_k = 0$ since $p_j p_k = p_k p_j$ is symmetric. The second term is proportional to L_z,

$$\left\{\vec{p} \times \vec{B} \cdot \vec{q}, L_z\right\} = \{B_z L_z, L_z\} = 0.$$

The final terms are proportional to q^2, and $\left\{q^2, \vec{q} \times \vec{p}\right\} = 0$ for the same reason that $\left\{p^2, \vec{q} \times \vec{p}\right\} = 0$, leading to

$$\{\mathcal{H}, \vec{q} \times \vec{p}\} = 0.$$

Hence angular momentum is conserved away from collisions as well.

(g) The most general form of the equilibrium distribution functions must set both the collision terms, and the streaming terms to zero. Based on the results of the previous parts, we thus obtain

$$f_\alpha = A_\alpha \exp\left[-\beta \mathcal{H}_\alpha - \gamma L_z\right].$$

The collision terms allow for the possibility of a term $-\vec{u} \cdot \vec{p}$ in the exponent, corresponding to an average velocity. Such a term will not commute with the potential set up by a stationary box, and is thus ruled out by the streaming terms. On the other hand, the angular momentum does commute with a circular potential $\{V(\vec{q}), L\} = 0$, and is allowed by the streaming terms. A non–zero γ describes the electron gas rotating in a circular box.

(h) Scattering from any impurity removes the conservation of $\vec{p}$, and hence $\vec{L}$, in collisions. The γ term will no longer be needed. Scattering from magnetic impurities mixes populations of up and down spins, necessitating $A_\uparrow = A_\downarrow$; non–magnetic impurities do not have this effect.

(i) Conservation of angular momentum is related to conservation of $\vec{p}$, as shown in (f), and hence does not lead to any new equation. In contrast, conservation of spin leads to an additional hydrodynamic equation involving the magnetization, which is proportional to $(n_\uparrow - n_\downarrow)$.

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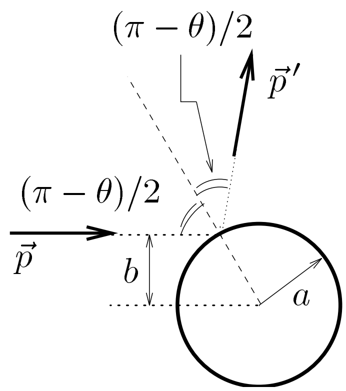
2. *The 2–dimensional Lorentz gas* is described by the Hamiltonian

$$\mathcal{H} = \sum_{i=1}^{N}\left[\frac{\vec{p}_i^{\,2}}{2m} + U(\vec{q}_i)\right],$$

where $U(\vec{q})$ describes collisions with a uniform density n_o of fixed hard circles of radius a.
(a) As in lectures, let b denote the impact parameter, which (see figure) is related to the angle θ between $\vec{p}\,'$ and $\vec{p}$ by



$$b(\theta) = a\sin\frac{\pi-\theta}{2} = a\cos\frac{\theta}{2}.$$

The differential cross section is then given by

$$d\sigma = 2|db| = a\sin\frac{\theta}{2}d\theta.$$

Hence the total cross section

$$\sigma_{tot} = \int_0^\pi d\theta a\sin\frac{\theta}{2} = 2a\left[-\cos\frac{\theta}{2}\right]_0^\pi = 2a.$$

(b) The corresponding Boltzmann equation is

$$\frac{\partial f}{\partial t} + \frac{\vec{p}}{m}\cdot\frac{\partial f}{\partial \vec{q}} + \vec{F}\cdot\frac{\partial f}{\partial \vec{p}}$$
$$= \int d\theta\frac{d\sigma}{d\theta}\frac{|\vec{p}|}{m}n_0\left[-f(\vec{p}\,)+f(\vec{p}\,')\right] = \frac{n_o|\vec{p}|}{m}\int d\theta\frac{d\sigma}{d\theta}\left[f(\vec{p}\,')-f(\vec{p})\right] \equiv C\left[f(\vec{p})\right].$$

(c) Using the definitions $\vec{F} \equiv -\partial U/\partial \vec{q}$,

$$n(\vec{q},t) = \int d^2\vec{p} f(\vec{q},\vec{p},t), \qquad \text{and} \qquad \langle g(\vec{q},t)\rangle = \frac{1}{n(\vec{q},t)} \int d^2\vec{p} f(\vec{q},\vec{p},t) g(\vec{q},t),$$

we can write

$$\begin{aligned}
\frac{d}{dt}\left(n\left\langle \chi(|\vec{p}\,|)\right\rangle\right) &= \int d^2p\chi(|\vec{p}\,|)\left[-\frac{\vec{p}}{m}\cdot\frac{\partial f}{\partial \vec{q}} - \vec{F}\cdot\frac{\partial f}{\partial \vec{p}} + \int d\theta \frac{d\sigma}{d\theta}\frac{|\vec{p}\,|}{m} n_o\left(f(\vec{p}) - f(\vec{p}\,')\right)\right] \\
&= -\frac{\partial}{\partial \vec{q}}\cdot\left(n\left\langle \frac{\vec{p}}{m}\chi\right\rangle\right) + \vec{F}\cdot\left(n\left\langle \frac{\partial \chi}{\partial \vec{p}}\right\rangle\right).
\end{aligned}$$

Rewriting this final expression gives the hydrodynamic equation

$$\frac{\partial}{\partial t}\left(n\langle\chi\rangle\right) + \frac{\partial}{\partial \vec{q}}\cdot\left(n\left\langle \frac{\vec{p}}{m}\chi\right\rangle\right) = \vec{F}\cdot\left(n\left\langle \frac{\partial \chi}{\partial \vec{p}}\right\rangle\right).$$

(d) Using $\chi = 1$ in the above expression

$$\frac{\partial}{\partial t} n + \frac{\partial}{\partial \vec{q}}\cdot\left(n\left\langle \frac{\vec{p}}{m}\right\rangle\right) = 0.$$

In terms of the local density $\rho = mn$, and velocity $\vec{u} \equiv \langle \vec{p}/m\rangle$, we have

$$\frac{\partial}{\partial t}\rho + \frac{\partial}{\partial \vec{q}}\cdot\left(\rho\vec{u}\,\right) = 0.$$

(e) With the kinetic energy $\chi = p^2/2m$ as a conserved quantity, the equation found in (c) gives

$$\frac{\partial}{\partial t}\left(\frac{n}{2m}\left\langle |\vec{p}\,|^2\right\rangle\right) + \frac{\partial}{\partial \vec{q}}\cdot\left(\frac{n}{2}\left\langle \frac{\vec{p}}{m}\frac{p^2}{m}\right\rangle\right) = \vec{F}\cdot\left(n\frac{\langle\vec{p}\rangle}{m}\right).$$

Substituting $\vec{p}/m = \vec{u} + \vec{c}$, where $\langle \vec{c}\,\rangle = 0$, and using $\rho = nm$,

$$\frac{\partial}{\partial t}\left[\frac{\rho}{2}u^2 + \frac{\rho}{2}\left\langle c^2\right\rangle\right] + \frac{\partial}{\partial \vec{q}}\cdot\left[\frac{\rho}{2}\left\langle(\vec{u}+\vec{c}\,)(u^2 + c^2 + 2\vec{u}\cdot\vec{c}\,)\right\rangle\right] = \frac{\rho}{m}\vec{F}\cdot\vec{u}.$$

From the definition $\varepsilon = \rho\left\langle c^2\right\rangle/2$, we have

$$\frac{\partial}{\partial t}\left[\frac{\rho}{2}u^2 + \varepsilon\right] + \frac{\partial}{\partial \vec{q}}\cdot\left[\frac{\rho}{2}\left(\vec{u}u^2 + \vec{u}\left\langle c^2\right\rangle + \left\langle \vec{c}c^2\right\rangle + 2\vec{u}\cdot\left\langle\vec{c}\,\vec{c}\,\right\rangle\right)\right] = \frac{\rho}{m}\vec{F}\cdot\vec{u}.$$

Finally, by substituting $\vec{h} \equiv \rho\left\langle \vec{c}\,c^2\right\rangle/2$ and $P_{\alpha\beta} = \rho\left\langle c_\alpha c_\beta\right\rangle$, we get

$$\frac{\partial}{\partial t}\left[\frac{\rho}{2}u^{\,2} + \varepsilon\right] + \frac{\partial}{\partial \vec{q}}\cdot\left[\vec{u}\left(\frac{\rho}{2}u^2 + \varepsilon\right) + \vec{h}\right] + \frac{\partial}{\partial q_\alpha}\left(u_\beta P_{\alpha\beta}\right) = \frac{\rho}{m}\vec{F}\cdot\vec{u}.$$

(f) There are only two quantities, 1 and $p^2/2m$, conserved in collisions. Let us start with the one particle density

$$f^0(\vec{p},\vec{q},t) = n(\vec{q},t)\exp\left[-\frac{p^2}{2mk_BT(\vec{q},t)}\right]\frac{1}{2\pi mk_BT(\vec{q},t)}.$$

Then

$$\vec{u} = \left\langle \frac{\vec{p}}{m} \right\rangle_0 = 0, \qquad \text{and} \qquad \vec{h} = \left\langle \frac{\vec{p}}{m}\frac{p^2}{m} \right\rangle_0 \frac{\rho}{2} = 0,$$

since both are odd functions of $\vec{p}$, while f^o is an even function of $\vec{p}$, while

$$P_{\alpha\beta} = \rho\left\langle c_\alpha c_\beta\right\rangle = \frac{n}{m}\left\langle p_\alpha p_\beta\right\rangle = \frac{n}{m}\delta\alpha\beta\cdot mk_BT = nk_BT\delta_{\alpha\beta}.$$

Substituting these expressions into the results for (c) and (d), we obtain the zeroth–order hydrodynamic equations

$$\begin{cases} \dfrac{\partial\rho}{\partial t} = 0, \\ \dfrac{\partial}{\partial t}\varepsilon = \dfrac{\partial}{\partial t}\dfrac{\rho}{2}\left\langle c^2\right\rangle = 0. \end{cases}$$

The above equations imply that ρ and ε are independent of time, i.e.

$$\rho = n(\vec{q}), \qquad \text{and} \qquad \varepsilon = k_BT(\vec{q}),$$

or

$$f^0 = \frac{n(\vec{q})}{2\pi mk_BT(\vec{q})}\exp\left[-\frac{p^2}{2mk_BT(\vec{q})}\right].$$

(g) The single collision time approximation is

$$C\left[f\right] = \frac{f^0 - f}{\tau}.$$

The first order solution to Boltzmann equation

$$f = f^0\left(1+g\right),$$

is obtained from

$$\mathcal{L}\left[f^0\right] = -\frac{f^0 g}{\tau},$$

as

$$g = -\tau\frac{1}{f^0}\mathcal{L}\left[f^0\right] = -\tau\left[\frac{\partial}{\partial t}\ln f^0 + \frac{\vec{p}}{m}\cdot\frac{\partial}{\partial\vec{q}}\ln f^0 + \vec{F}\cdot\frac{\partial}{\partial\vec{p}}\ln f^0\right].$$

Noting that

$$\ln f^0 = -\frac{p^2}{2mk_BT} + \ln n - \ln T - \ln(2\pi mk_B),$$

where n and T are independent of t, we have $\partial \ln f^0/\partial t = 0$, and

$$\begin{aligned} g &= -\tau\left\{\vec{F}\cdot\left(\frac{-\vec{p}}{mk_BT}\right) + \frac{\vec{p}}{m}\cdot\left[\frac{1}{n}\frac{\partial n}{\partial \vec{q}} - \frac{1}{T}\frac{\partial T}{\partial \vec{q}} + \frac{p^2}{2mk_BT^2}\frac{\partial T}{\partial \vec{q}}\right]\right\} \\ &= -\tau\left\{\frac{\vec{p}}{m}\cdot\left(\frac{1}{\rho}\frac{\partial \rho}{\partial \vec{q}} - \frac{1}{T}\frac{\partial T}{\partial \vec{q}} + \frac{p^2}{2mk_BT^2}\frac{\partial T}{\partial \vec{q}} - \frac{\vec{F}}{k_BT}\right)\right\}. \end{aligned}$$

(h) Clearly $\int d^2q f^0\,(1+g) = \int d^2q f^0 = n$, and

$$\begin{aligned} u_\alpha &= \left\langle\frac{p_\alpha}{m}\right\rangle = \frac{1}{n}\int d^2p \frac{p_\alpha}{m} f^0(1+g) \\ &= \frac{1}{n}\int d^2p \frac{p_\alpha}{m}\left[-\tau\frac{p_\beta}{m}\left(\frac{\partial \ln\rho}{\partial q_\beta} - \frac{\partial \ln T}{\partial q_\beta} - \frac{F_\beta}{k_BT} + \frac{p^2}{2mk_BT^2}\frac{\partial T}{\partial q_\beta}\right)\right] f^0. \end{aligned}$$

Wick's theorem can be used to check that

$$\begin{aligned} \langle p_\alpha p_\beta\rangle_0 &= \delta_{\alpha\beta} mk_BT, \\ \left\langle p^2 p_\alpha p_\beta\right\rangle_0 &= (mk_BT)^2\left[2\delta_{\alpha\beta} + 2\delta_{\alpha\beta}\right] = 4\delta_{\alpha\beta}(mk_BT)^2, \end{aligned}$$

resulting in

$$u_\alpha = -\frac{n\tau}{\rho}\left[\delta_{\alpha\beta}k_BT\left(\frac{\partial}{\partial q_\beta}\ln\left(\frac{\rho}{T}\right) - \frac{1}{k_BT}F_\beta\right) + 2k_B\frac{\partial T}{\partial q_\beta}\delta_{\alpha\beta}\right].$$

Rearranging these terms yields

$$\rho u_\alpha = n\tau\left[F_\alpha - k_BT\frac{\partial}{\partial q_\alpha}\ln(\rho T)\right].$$

(i) The velocity response function is now calculated easily as

$$\chi_{\alpha\beta} = \frac{\partial u_\alpha}{\partial F_\beta} = \frac{n\tau}{\rho}\delta_{\alpha\beta}.$$

(j) The first order expressions for pressure tensor and heat flux are

$$\begin{aligned} P_{\alpha\beta} &= \frac{\rho}{m^2}\langle p_\alpha p_\beta\rangle = \delta_{\alpha\beta} nk_BT, \qquad \text{and} \qquad \delta^1 P_{\alpha\beta} = 0, \\ h_\alpha &= \frac{\rho}{2m^3}\left\langle p_\alpha p^2\right\rangle = -\frac{\tau\rho}{2m^3}\left\langle p_\alpha p^2\frac{p_i}{m}\left(a_i + b_i p^2\right)\right\rangle_0. \end{aligned}$$

The latter is calculated from Wick's theorem results

$$\left\langle p_i p_\alpha p^2 \right\rangle = 4\delta_{\alpha i} \left(m k_B T\right)^2, \qquad \text{and}$$
$$\left\langle p_i p_\alpha p^2 p^2 \right\rangle = \left(m k_B T\right)^3 \left[\delta_{\alpha i}(4+2) + 4 \times 2\delta\alpha i + 4 \times 2\delta_{\alpha i}\right] = 22\delta_{\alpha i},$$

as

$$\begin{aligned} h_\alpha &= -\frac{\rho\tau}{2m^3}\left[\left(m k_B T\right)^2\left(\frac{\partial}{\partial q_\alpha}\ln\frac{\rho}{T} - \vec{F}\right) + \frac{22(m k_B T)^3}{2m k_b T}\frac{\partial}{\partial q_\alpha}\ln T\right] \\ &= -11 n k_B^2 T\tau\frac{\partial T}{\partial q_\alpha}. \end{aligned}$$

Substitute these expressions for $P_{\alpha\beta}$ and h_α into the equation obtained in (e)

$$\frac{\partial}{\partial t}\left[\frac{\rho}{2}u^2 + \epsilon\right] + \frac{\partial}{\partial \vec{q}} \cdot \left[\vec{u}\left(\frac{\rho}{2}u^2 + \epsilon\right) - 11 n k_B^2 T\tau\frac{\partial T}{\partial \vec{q}}\right] + \frac{\partial}{\partial \vec{q}}\left(\vec{u}\, n k_B T\right) = \frac{\rho}{m}\vec{F} \cdot \vec{u}.$$

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