Review Problems

The final exam will take place on **Tuesday December 16**, in the **Johnson Athletic Center**, from **1:30 to 4:30pm**. All topics presented in the course will be covered, with emphasis on the second half. It will be a closed book exam, but you may bring a two–sided sheet of formulas if you wish. It may also be helpful to bring along a calculator. There will be a lecture on Monday **12/8/03**, and a recitation with quiz review on Wednesday **12/10/03**.

The enclosed exams (and solutions) from the previous years are intended to help you review the material.

****

Note that the first parts of each problem are easier than its last parts. Therefore, make sure to proceed to the next problem when you get stuck.

You may find the following information helpful:

Physical Constants

Electron mass	$m_e \approx 9.1 \times 10^{-31} Kg$	Proton mass	$m_p \approx 1.7 \times 10^{-27} Kg$
Electron Charge	$e \approx 1.6 \times 10^{-19} C$	Planck's constant/2π	$\hbar \approx 1.1 \times 10^{-34} Js^{-1}$
Speed of light	$c \approx 3.0 \times 10^{8} ms^{-1}$	Stefan's constant	$\sigma \approx 5.7 \times 10^{-8} Wm^{-2}K^{-4}$
Boltzmann's constant	$k_B \approx 1.4 \times 10^{-23} JK^{-1}$	Avogadro's number	$N_0 \approx 6.0 \times 10^{23} mol^{-1}$

Conversion Factors

$1atm \equiv 1.0 \times 10^5 Nm^{-2}$ $\quad$ $1\mathring{A} \equiv 10^{-10} m$ $\quad$ $1eV \equiv 1.1 \times 10^4 K$

Thermodynamics

$dE = TdS + dW$ $\quad$ For a gas: $dW = -PdV$ $\quad$ For a film: $dW = \sigma dA$

Mathematical Formulas

$\lim_{x\to\infty} \coth x = 1 + 2e^{-2x} + \mathcal{O}\left(e^{-4x}\right)$ $\quad$ $\lim_{x\to 0} \coth x = \frac{1}{x} + \frac{x}{3} + \mathcal{O}\left(x^2\right)$

$\int_0^\infty dx\ x^n\ e^{-\alpha x} = \frac{n!}{\alpha^{n+1}}$ $\quad$ $(\frac{1}{2})! = \frac{\sqrt{\pi}}{2}$

$\int_{-\infty}^{\infty} dx \exp\left[-ikx - \frac{x^2}{2\sigma^2}\right] = \sqrt{2\pi\sigma^2} \exp\left[-\frac{\sigma^2 k^2}{2}\right]$ $\quad$ $\lim_{N\to\infty} \ln N! = N \ln N - N$

$\left\langle e^{-ikx} \right\rangle = \sum_{n=1}^{\infty} \frac{(-ik)^n}{n!} \left\langle x^n \right\rangle$ $\quad$ $\ln \left\langle e^{-ikx} \right\rangle = \sum_{n=1}^{\infty} \frac{(-ik)^n}{n!} \left\langle x^n \right\rangle_c$

$f_m^\eta(z) = \frac{1}{(m-1)!} \int_0^\infty dx \frac{x^{m-1}}{z^{-1}e^x - \eta} = \sum_{\alpha=1}^{\infty} \eta^{\alpha+1} \frac{z^\alpha}{\alpha^m}$ $\quad$ $\frac{df_m^\eta}{dz} = \frac{1}{z} f_{m-1}^\eta$

$\lim_{z\to\infty} f_m^-(z) = \frac{(\ln z)^m}{m!} \left[1 + \frac{\pi^2}{6} m(m-1)(\ln z)^{-2} + \cdots\right]$ $\quad$ $f_2^-(1) = \frac{\pi^2}{12}$ $\quad$ $f_4^-(1) = \frac{7\pi^4}{720}$

$\zeta_m \equiv f_m^+(1)$ $\quad$ $\zeta_{3/2} \approx 2.612$ $\quad$ $\zeta_2 = \frac{\pi^2}{6}$ $\quad$ $\zeta_{5/2} \approx 1.341$ $\quad$ $\zeta_3 \approx 1.202$ $\quad$ $\zeta_4 = \frac{\pi^4}{90}$

1. *Exciton dissociation in a semiconductor:* By shining an intense laser beam on a semiconductor, one can create a metastable collection of electrons (charge $-e$, and effective mass $m_{\rm e}$) and holes (charge $+e$, and effective mass $m_{\rm h}$) in the bulk. The oppositely charged particles may pair up (as in a hydrogen atom) to form a gas of *excitons*, or they may dissociate into a plasma. We shall examine a much simplified model of this process.

(a) Calculate the free energy of a gas composed of $N_{\rm e}$ electrons and $N_{\rm h}$ holes, at temperature T, treating them as classical non-interacting particles of masses $m_{\rm e}$ and $m_{\rm h}$.

(b) By pairing into an excition, the electron hole pair lowers its energy by ε. [The binding energy of a hydrogen-like exciton is $\varepsilon \approx me^4/(2\hbar^2\epsilon^2)$, where ϵ is the dielectric constant, and $m^{-1} = m_{\rm e}^{-1} + m_{\rm h}^{-1}$.] Calculate the free energy of a gas of $N_{\rm p}$ excitons, treating them as classical non-interacting particles of mass $m = m_{\rm e} + m_{\rm h}$.

(c) Calculate the chemical potentials $\mu_{\rm e}$, $\mu_{\rm h}$, and $\mu_{\rm p}$ of the electron, hole, and exciton states, respectively.

(d) Express the equilibrium condition between excitons and electron/holes in terms of their chemical potentials.

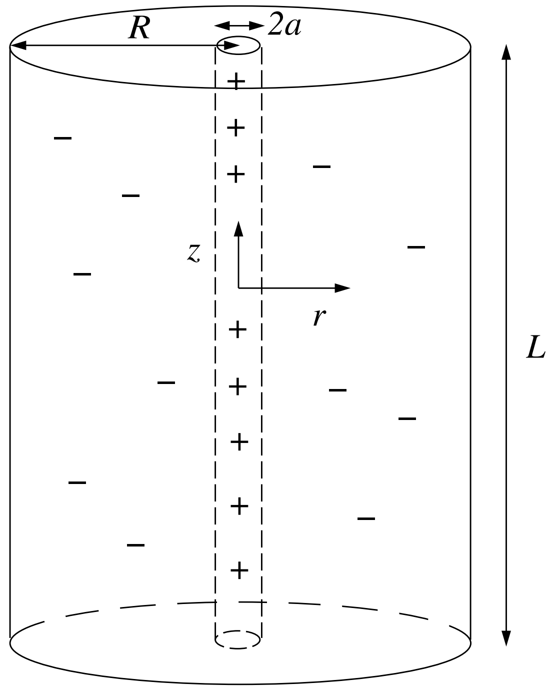
(e) At a high temperature T, find the density $n_{\rm p}$ of excitons, as a function of the total density of excitations $n \approx n_{\rm e} + n_{\rm h}$.

****

2. *The Manning Transition:* When ionic polymers (polyelectrolytes) such as DNA are immersed in water, the negatively charged *counter-ions* go into solution, leaving behind a positively charged polymer. Because of the electrostatic repulsion of the charges left behind, the polymer stretches out into a cylinder of radius a, as illustrated in the figure. While thermal fluctuations tend to make the ions wander about in the solvent, electrostatic attractions favor their return and condensation on the polymer. If the number of counter-ions is N, they interact with the N positive charges left behind on the rod through the potential $U(r) = -2(Ne/L)\ln(r/L)$, where r is the radial coordinate in a cylindrical geometry. If we ignore the Coulomb repulsion between counter-ions, they can be described by the classical Hamiltonian

$$\mathcal{H} = \sum_{i=1}^{N} \left[\frac{p_i^2}{2m} + 2e^2 n \ln\left(\frac{r}{L}\right) \right],$$

where $n = N/L$.



(a) For a cylindrical container of radius R, calculate the canonical partition function Z in terms of temperature T, density n, and radii R and a.

(b) Calculate the probability distribution function $p(r)$ for the radial position of a counter-ion, and its first moment $\langle r \rangle$, the average radial position of a counter-ion.

(c) The behavior of the results calculated above in the limit $R \gg a$ is very different at high and low temperatures. Identify the transition temperature, and characterize the nature of the two phases. In particular, how does $\langle r \rangle$ depend on R and a in each case?

(d) Calculate the pressure exerted by the counter-ions on the wall of the container, at $r = R$, in the limit $R \gg a$, at all temperatures.

(e) The character of the transition examined in part (d) is modified if the Coulomb interactions between counter-ions are taken into account. An approximate approach to the interacting problem is to allow a fraction N_1 of counter-ions to condense along the polymer rod, while the remaining $N_2 = N - N_1$ fluctuate in the solvent. The free counter-ions are again treated as non-interacting particles, governed by the Hamiltonian

$$\mathcal{H} = \sum_{i=1}^{N} \left[\frac{p_i^2}{2m} + 2e^2 n_2 \ln\left(\frac{r}{L}\right) \right],$$

where $n_2 = N_2/L$. *Guess* the equilibrium number of non-interacting ions, N_2^*, and justify your guess by discussing the response of the system to slight deviations from N_2^*. (This is a qualitative question for which no new calculations are needed.)

****

3. *Bose gas in d dimensions:* Consider a gas of non-interacting (spinless) bosons with an energy spectrum $\epsilon = p^2/2m$, contained in a box of "volume" $V = L^d$ in d dimensions.

(a) Calculate the grand potential $\mathcal{G} = -k_\mathrm{B}T \ln \mathcal{Q}$, and the density $n = N/V$, at a chemical potential μ. Express your answers in terms of d and $f_m^+(z)$, where $z = e^{\beta\mu}$, and

$$f_m^+(z) = \frac{1}{\Gamma(m)} \int_0^\infty \frac{x^{m-1}}{z^{-1}e^x - 1} dx.$$

(Hint: Use integration by parts on the expression for $\ln \mathcal{Q}$.)

(b) Calculate the ratio PV/E, and compare it to the classical value.

(c) Find the critical temperature, $T_\mathrm{c}(n)$, for Bose-Einstein condensation.

(d) Calculate the heat capacity $C(T)$ for $T < T_\mathrm{c}(n)$.

(e) Sketch the heat capacity at all temperatures.

(f) Find the ratio, $C_\mathrm{max}/C(T \to \infty)$, of the maximum heat capacity to its classical limit, and evaluate it in $d = 3$

(g) How does the above calculated ratio behave as $d \to 2$? In what dimensions are your results valid? Explain.

****

1. *Electron Magnetism:* The conduction electrons in a metal can be treated as a gas of fermions of spin 1/2 (with up/down degeneracy), and density $n = N/V$.

(a) Ignoring the interactions between electrons, describe (in words) their ground state. Calculate the fermi wave number $k_{\rm F}$, and the ground-state energy density E_0/V in terms of the density n.

Electrons also interact *via* the Coulomb repulsion, which favors a wave function which is antisymmetric in position space, thus keeping them apart. Because of the full (position *and* spin) antisymmetry of fermionic wave functions, this interaction may be described as an effective spin-spin coupling which favors states with parallel spins. In a simple approximation, the effect of this interaction is represented by adding a potential

$$U = \alpha \frac{N_+ N_-}{V},$$

to the Hamiltonian, where N_+ and $N_- = N - N_+$ are the numbers of electrons with up and down spins, and V is the volume. (The parameter α is related to the scattering length a by $\alpha = 4\pi\hbar^2 a/m$.) We would like to find out if the unmagnetized gas with $N_+ = N_- = N/2$ still minimizes the energy, or if the gas is spontaneously magnetized.

(b) Express the modified Fermi wave numbers $k_{\rm F+}$ and $k_{\rm F-}$, in terms of the densities $n_+ = N_+/V$ and $n_- = N_-/V$.

(c) Assuming small deviations $n_+ = n/2 + \delta$ and $n_- = n/2 - \delta$ from the symmetric state, calculate the change in the kinetic energy of the system to second order in δ.

(d) Express the spin-spin interaction density in terms of δ. Find the critical value of α_c, such that for $\alpha > \alpha_c$ the electron gas can lower its total energy by spontaneously developing a magnetization. (This is known as the *Stoner instability.*)

(e) Explain qualitatively, and sketch the behavior of the spontaneous magnetization as a function of α.

****

2. *Boson magnetism:* Consider a gas of non-interacting spin 1 bosons, each subject to a Hamiltonian

$$\mathcal{H}_1(\vec{p}, s_z) = \frac{\vec{p}^{\,2}}{2m} - \mu_0 s_z B \quad ,$$

where $\mu_0 = e\hbar/mc$, and s_z takes *three* possible values of (-1, 0, +1). (The orbital effect, $\vec{p} \to \vec{p} - e\vec{A}$, has been ignored.)

(a) In a grand canonical ensemble of chemical potential μ, what are the average occupation numbers $\left\{ \langle n_+(\vec{k})\rangle, \langle n_0(\vec{k})\rangle, \langle n_-(\vec{k})\rangle \right\}$, of one-particle states of wavenumber $\vec{k} = \vec{p}/\hbar$?

(b) Calculate the average total numbers $\{N_+, N_0, N_-\}$, of bosons with the three possible values of s_z in terms of the functions $f_m^+(z)$.

(c) Write down the expression for the magnetization $M(T,\mu) = \mu_0(N_+ - N_-)$, and by expanding the result for small B find the *zero field susceptibility* $\chi(T,\mu) = \partial M/\partial B|_{B=0}$.

To find the behavior of $\chi(T,n)$, where $n = N/V$ is the total density, proceed as follows:

(d) For $B = 0$, find the high temperature expansion for $z(\beta,n) = e^{\beta\mu}$, correct to second order in n. Hence obtain the first correction from quantum statistics to $\chi(T,n)$ at high temperatures.

(e) Find the temperature $T_c(n, B = 0)$, of Bose-Einstein condensation. What happens to $\chi(T,n)$ on approaching $T_c(n)$ from the high temperature side?

(f) What is the chemical potential μ for $T < T_c(n)$, at a small but finite value of B? Which one-particle state has a macroscopic occupation number?

(g) Using the result in (f), find the spontaneous magnetization,

$$\overline{M}(T,n) = \lim_{B\to 0} M(T,n,B).$$

****

3. *The virial theorem* is a consequence of the invariance of the phase space for a system of N (classical or quantum) particles under canonical transformations, such as a change of scale. In the following, consider N particles with coordinates $\{\vec{q}_i\}$, and conjugate momenta $\{\vec{p}_i\}$ (with $i = 1, \cdots, N$), and subject to a Hamiltonian $\mathcal{H}(\{\vec{p}_i\}, \{\vec{q}_i\})$.

(a) *Classical version*: Write down the expression for classical partition function, $Z \equiv Z[\mathcal{H}]$. Show that it is invariant under the rescaling $\vec{q}_1 \to \lambda\vec{q}_1$, $\vec{p}_1 \to \vec{p}_1/\lambda$ of a pair of conjugate variables, i.e. $Z[\mathcal{H}_\lambda]$ is independent of λ, where $\mathcal{H}_\lambda$ is the Hamiltonian obtained after the above rescaling.

(b) *Quantum mechanical version*: Write down the expression for quantum partition function. Show that it is also invariant under the rescalings $\vec{q}_1 \to \lambda\vec{q}_1$, $\vec{p}_1 \to \vec{p}_1/\lambda$, where $\vec{p}_i$ and $\vec{q}_i$ are now quantum mechanical operators. (Hint: start with the time-independent Schrödinger equation.)

(c) Now assume a Hamiltonian of the form

$$\mathcal{H} = \sum_i \frac{\vec{p}_i^{\,2}}{2m} + V(\{\vec{q}_i\}).$$

Use the result that $Z[\mathcal{H}_\lambda]$ is independent of λ to prove the *virial* relation

$$\left\langle \frac{\vec{p}_1^{\,2}}{m} \right\rangle = \left\langle \frac{\partial V}{\partial \vec{q}_1} \cdot \vec{q}_1 \right\rangle,$$

where the brackets denote thermal averages. (You may formulate your answer in the classical language, as a possible quantum derivation is similar.)

(d) The above relation is sometimes used to estimate the mass of distant galaxies. The stars on the outer boundary of the G-8.333 galaxy have been measured to move with velocity $v \approx 200$ km/s. Give a numerical estimate of the ratio of the G-8.333's mass to its size.

****

1. *Freezing of He^3:* At low temperatures He^3 can be converted from liquid to solid by application of pressure. A peculiar feature of its phase boundary is that $(dP/dT)_{\rm melting}$ is negative at temperatures below $0.3\,^oK$ $[(dP/dT)_m \approx -30{\rm atm}\ ^oK^{-1}$ at $T \approx 0.1\,^oK]$. We will use a simple model of liquid and solid phases of He^3 to account for this feature.

(a) In the solid phase, the He^3 atoms form a crystal lattice. Each atom has nuclear spin of 1/2. Ignoring the interaction between spins, what is the entropy per particle s_s, due to the spin degrees of freedom?

(b) Liquid He^3 is modelled as an ideal Fermi gas, with a volume of $46\mathring{A}^3$ per atom. What is its Fermi temperature T_F, in degrees Kelvin?

(c) How does the heat capacity of liquid He^3 behave at low temperatures? Write down an expression for C_V in terms of N, T, k_B, T_F, *up to a numerical constant*, that is valid for $T \ll T_F$.

(d) Using the result in (c), calculate the entropy per particle s_ℓ, in the liquid at low temperatures. For $T \ll T_F$, which phase (solid or liquid) has the higher entropy?

(e) By equating chemical potentials, or by any other technique, prove the Clausius–Clapeyron equation $(dP/dT)_{\rm melting} = (s_\ell - s_s)/(v_\ell - v_s)$, where v_ℓ and v_s are the volumes per particle in the liquid and solid phases respectively.

(f) It is found experimentally that $v_\ell - v_s = 3\mathring{A}^3$ per atom. Using this information, plus the results obtained in previous parts, estimate $(dP/dT)_{\rm melting}$ at $T \ll T_F$.

****

2. *Non-interacting bosons:* Consider a grand canonical ensemble of non-interacting *bosons* with chemical potential μ. The one–particle states are labelled by a wavevector $\vec{q}$, and have energies $\mathcal{E}(\vec{q})$.

(a) What is the joint probability $P(\{n_{\vec{q}}\})$, of finding a set of occupation numbers $\{n_{\vec{q}}\}$, of the one–particle states, in terms of the fugacities $z_{\vec{q}} \equiv \exp\left[\beta(\mu - \mathcal{E}(\vec{q}))\right]$?

(b) For a particular $\vec{q}$, calculate the characteristic function $\langle \exp\left[ikn_{\vec{q}}\right]\rangle$.

(c) Using the result of part (b), **or otherwise**, give expressions for the mean and variance of $n_{\vec{q}}$. occupation number $\langle n_{\vec{q}}\rangle$.

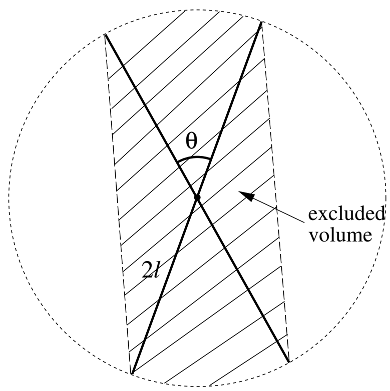
(d) Express the variance in part (c) in terms of the mean occupation number $\langle n_{\vec{q}}\rangle$.

(e) Express your answer to part (a) in terms of the occupation numbers $\{\langle n_{\vec{q}}\rangle\}$.

(f) Calculate the entropy of the probability distribution for bosons, in terms of $\{\langle n_{\vec{q}}\rangle\}$, and comment on its zero temperature limit.

****

3. *Hard rods:* A collection of N asymmetric molecules in two dimensions may be modeled as a gas of rods, each of length $2l$ and lying in a plane. A rod can move by translation of its center of mass and rotation about latter, as long as it does not encounter another rod. Without treating the hard-core interaction exactly, we can incorporate it approximately by assuming that the rotational motion of each rod is restricted (by the other rods) to an angle θ, which in turn introduces an excluded volume $\Omega(\theta)$ (associated with each rod). The value of θ is then calculated self consistently by maximizing the entropy at a given density $n = N/V$, where V is the total accessible area.



(a) Write down the entropy of such a collection of rods in terms of N, n, Ω, and $A(\theta)$, the entropy associated to the rotational freedom of a *single* rod. (You may ignore the momentum contributions throughout, and consider the large N limit.)

(b) Extremizing the entropy as a function of θ, relate the density to Ω, A, and their derivatives Ω', A'; express your result in the form $n = f(\Omega, A, \Omega', A')$.

(c) Express the excluded volume Ω in terms of θ and sketch f as a function of $\theta \in [0, \pi]$, assuming $A \propto \theta$.

(d) Describe the equilibrium state at high densities. Can you identify a phase transition as the density is decreased? Draw the corresponding critical density n_c on your sketch. What is the critical angle θ_c at the transition? You don't need to calculate θ_c explicitly, but give an (implicit) relation defining it. What value does θ adopt at $n < n_c$?

****