Review Problems

The Mid-term quiz will take place on Monday **10/24/05** in room **1-190** from **2:30 to 4:00** pm. There will be a recitation with quiz review on Friday **10/21/05**.

All topics up to (but not including) the micro-canonical ensemble will be covered. The exam is 'closed book,' but if you wish you may bring a two-sided sheet of formulas. The enclosed exams (and solutions) from the previous years are intended to help you review the material. *Solutions to the midterm are available in the exams section.*

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Answer all three problems, but note that the first parts of each problem are easier than its last parts. Therefore, make sure to proceed to the next problem when you get stuck.

You may find the following information helpful:

Physical Constants

Electron mass	$m_e \approx 9.1 \times 10^{-31} Kg$	Proton mass	$m_p \approx 1.7 \times 10^{-27} Kg$
Electron Charge	$e \approx 1.6 \times 10^{-19} C$	Planck's const./2π	$\hbar \approx 1.1 \times 10^{-34} Js^{-1}$
Speed of light	$c \approx 3.0 \times 10^{8} ms^{-1}$	Stefan's const.	$\sigma \approx 5.7 \times 10^{-8} Wm^{-2}K^{-4}$
Boltzmann's const.	$k_B \approx 1.4 \times 10^{-23} JK^{-1}$	Avogadro's number	$N_0 \approx 6.0 \times 10^{23} mol^{-1}$

Conversion Factors

$1atm \equiv 1.0 \times 10^5 Nm^{-2}$ $\quad$ $1\mathring{A} \equiv 10^{-10} m$ $\quad$ $1eV \equiv 1.1 \times 10^4 K$

Thermodynamics

$dE = TdS + đW$ $\quad$ For a gas: $đW = -PdV$ $\quad$ For a wire: $đW = Jdx$

Mathematical Formulas

$\int_0^\infty dx\, x^n\, e^{-\alpha x} = \frac{n!}{\alpha^{n+1}}$ $\quad$ $\left(\frac{1}{2}\right)! = \frac{\sqrt{\pi}}{2}$

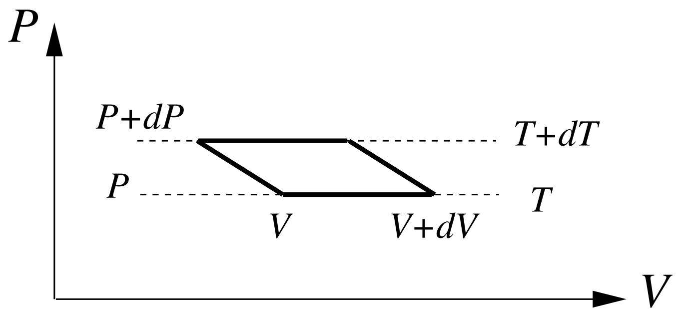
$\int_{-\infty}^{\infty} dx \exp\left[-ikx - \frac{x^2}{2\sigma^2}\right] = \sqrt{2\pi\sigma^2} \exp\left[-\frac{\sigma^2 k^2}{2}\right]$ $\quad$ $\lim_{N\to\infty} \ln N! = N \ln N - N$

$\left\langle e^{-ikx} \right\rangle = \sum_{n=0}^{\infty} \frac{(-ik)^n}{n!} \left\langle x^n \right\rangle$ $\quad$ $\ln \left\langle e^{-ikx} \right\rangle = \sum_{n=1}^{\infty} \frac{(-ik)^n}{n!} \left\langle x^n \right\rangle_c$

$\cosh(x) = 1 + \frac{x^2}{2!} + \frac{x^4}{4!} + \cdots$ $\quad$ $\sinh(x) = x + \frac{x^3}{3!} + \frac{x^5}{5!} + \cdots$

Surface area of a unit sphere in d dimensions $\quad$ $S_d = \frac{2\pi^{d/2}}{(d/2-1)!}$

1. *Photon gas Carnot cycle:* The aim of this problem is to obtain the blackbody radiation relation, $E(T,V) \propto VT^4$, starting from the equation of state, by performing an infinitesimal Carnot cycle on the photon gas.



(a) Express the work done, W, in the above cycle, in terms of dV and dP.

(b) Express the heat absorbed, Q, in expanding the gas *along an isotherm*, in terms of P, dV, and an appropriate derivative of $E(T,V)$.

(c) Using the efficiency of the Carnot cycle, relate the above expressions for W and Q to T and dT.

(d) Observations indicate that the pressure of the photon gas is given by $P = AT^4$, where $A = \pi^2 k_B^4/45\,(\hbar c)^3$ is a constant. Use this information to obtain $E(T,V)$, assuming $E(0,V) = 0$.

(e) Find the relation describing the *adiabatic paths* in the above cycle.

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2. *Moments of momentum:* Consider a gas of N classical particles of mass m in thermal equilibrium at a temperature T, in a box of volume V.

(a) Write down the equilibrium one particle density $f_{\text{eq.}}(\vec{p},\vec{q})$, for coordinate $\vec{q}$, and momentum $\vec{p}$.

(b) Calculate the joint characteristic function, $\left\langle \exp\left(-i\vec{k}\cdot\vec{p}\right)\right\rangle$, for momentum.

(c) Find all the joint cumulants $\left\langle p_x^\ell p_y^m p_z^n \right\rangle_c$.

(d) Calculate the joint moment $\langle p_\alpha p_\beta \left(\vec{p}\cdot\vec{p}\right)\rangle$.

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3. *Light and matter:* In this problem we use kinetic theory to explore the equilibrium between atoms and radiation.

(a) The atoms are assumed to be either in their ground state a_0, or in an excited state a_1, which has a higher energy ε. By considering the atoms as a collection of N fixed two-state systems of energy E (i.e. ignoring their coordinates and momenta), calculate the ratio n_1/n_0 of densities of atoms in the two states as a function of temperature T.

Consider photons γ of frequency $\omega = \varepsilon/\hbar$ and momentum $|\vec{p}| = \hbar\omega/c$, which can interact with the atoms through the following processes:

(i) *Spontaneous emission:* $a_1 \to a_0 + \gamma$.

(ii) *Adsorption:* $a_0 + \gamma \to a_1$.

(iii) *Stimulated emission:* $a_1 + \gamma \to a_0 + \gamma + \gamma$.

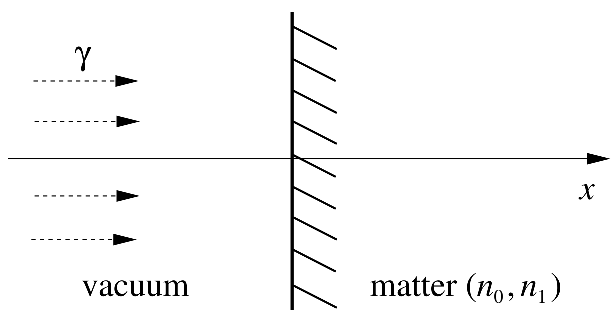
Assume that spontaneous emission occurs with a probability $\sigma_{\rm sp}$, and that adsorption and stimulated emission have constant (angle-independent) differential cross-sections of $\sigma_{\rm ad}/4\pi$ and $\sigma_{\rm st}/4\pi$, respectively.

(b) Write down the Boltzmann equation governing the density f of the photon gas, treating the atoms as fixed scatterers of densities n_0 and n_1.

(c) Find the equilibrium density $f_{\rm eq.}$ for the photons of the above frequency.

(d) According to Planck's law, the density of photons at a temperature T depends on their frequency ω as $f_{\rm eq.} = \left[\exp\left(\hbar\omega/k_BT\right) - 1\right]^{-1}/h^3$. What does this imply about the above cross sections?

(e) Consider a situation in which light shines along the x axis on a collection of atoms whose boundary coincides with the $x = 0$ plane, as illustrated in the figure.



Clearly, f will depend on x (and p_x), but will be independent of y and z. Adapt the Boltzmann equation you propose in part (b) to the case of a uniform incoming flux of photons with momentum $\vec{p} = \hbar\omega\hat{x}/c$. What is the *penetration length* across which the incoming flux decays?

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1. *Superconducting transition:* Many metals become superconductors at low temperatures T, and magnetic fields B. The heat capacities of the two phases at zero magnetic field are approximately given by

$$\begin{cases} C_s(T) = V\alpha T^3 & \text{in the superconducting phase} \\ C_n(T) = V\left[\beta T^3 + \gamma T\right] & \text{in the normal phase} \end{cases},$$

where V is the volume, and $\{\alpha, \beta, \gamma\}$ are constants. (There is no appreciable change in volume at this transition, and mechanical work can be ignored throughout this problem.)

(a) Calculate the entropies $S_s(T)$ and $S_n(T)$ of the two phases at zero field, using the third law of thermodynamics.

(b) Experiments indicate that there is no latent heat ($L = 0$) for the transition between the normal and superconducting phases at zero field. Use this information to obtain the transition temperature T_c, as a function of α, β, and γ.

(c) At zero temperature, the electrons in the superconductor form bound Cooper pairs. As a result, the internal energy of the superconductor is reduced by an amount $V\Delta$, i.e. $E_n(T = 0) = E_0$ and $E_s(T = 0) = E_0 - V\Delta$ for the metal and superconductor, respectively. Calculate the internal energies of both phases at finite temperatures.

(d) By comparing the Gibbs free energies (or chemical potentials) in the two phases, obtain an expression for the energy gap Δ in terms of α, β, and γ.

(e) In the presence of a magnetic field B, inclusion of magnetic work results in $dE = TdS + BdM + \mu dN$, where M is the magnetization. The superconducting phase is a perfect diamagnet, expelling the magnetic field from its interior, such that $M_s = -VB/(4\pi)$ in appropriate units. The normal metal can be regarded as approximately non-magnetic, with $M_n = 0$. Use this information, in conjunction with previous results, to show that the superconducting phase becomes normal for magnetic fields larger than

$$B_c(T) = B_0\left(1 - \frac{T^2}{T_c^2}\right),$$

giving an expression for B_0.

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2. *Probabilities:* Particles of type A or B are chosen independently with probabilities p_A and p_B.

(a) What is the probability $p(N_A, N)$ that N_A out of the N particles are of type A?

(b) Calculate the mean and the variance of N_A.

(c) Use the central limit theorem to obtain the probability $p(N_A, N)$ for large N.

(d) Apply Stirling's approximation ($\ln N! \approx N \ln N - N$) to $\ln p(N_A, N)$ [using the probability calculated in part (a), **not** part (c)] to find the most likely value, $\overline{N_A}$, for $N \gg 1$.

(e) Expand $\ln p(N_A, N)$ calculated in (d) around its maximum to second order in $\left(N_A - \overline{N_A}\right)$, and check for consistency with the result from the central limit theorem.

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3. *Thermal Conductivity:* Consider a classical gas between two plates separated by a distance w. One plate at $y = 0$ is maintained at a temperature T_1, while the other plate at $y = w$ is at a different temperature T_2. The gas velocity is zero, so that the initial zeroth order approximation to the one particle density is,

$$f_1^0(\vec{p}, x, y, z) = \frac{n(y)}{\left[2\pi m k_B T(y)\right]^{3/2}} \exp\left[-\frac{\vec{p}\cdot\vec{p}}{2mk_B T(y)}\right].$$

(a) What is the necessary relation between $n(y)$ and $T(y)$, to ensure that the gas velocity $\vec{u}$ remains zero? (Use this relation between $n(y)$ and $T(y)$ in the remainder of this problem.)

(b) Using Wick's theorem, or otherwise, show that

$$\left\langle p^2\right\rangle^0 \equiv \left\langle p_\alpha p_\alpha\right\rangle^0 = 3\left(mk_BT\right), \quad \text{and} \quad \left\langle p^4\right\rangle^0 \equiv \left\langle p_\alpha p_\alpha p_\beta p_\beta\right\rangle^0 = 15\left(mk_BT\right)^2,$$

where $\langle \mathcal{O}\rangle^0$ indicates local averages with the Gaussian weight f_1^0. Use the result $\left\langle p^6\right\rangle^0 = 105(mk_BT)^3$ (*you don't have to derive this*) in conjunction with symmetry arguments to conclude

$$\left\langle p_y^2 p^4\right\rangle^0 = 35\left(mk_BT\right)^3.$$

(c) The zeroth order approximation does not lead to relaxation of temperature/density variations related as in part (a). Find a better (time independent) approximation $f_1^1(\vec{p}, y)$, by linearizing the Boltzmann equation in the single collision time approximation, to

$$\mathcal{L}\left[f_1^1\right] \approx \left[\frac{\partial}{\partial t} + \frac{p_y}{m}\frac{\partial}{\partial y}\right] f_1^0 \approx -\frac{f_1^1 - f_1^0}{\tau_K},$$

where τ_K is of the order of the mean time between collisions.

(d) Use f_1^1, along with the averages obtained in part (b), to calculate h_y, the y component of the heat transfer vector, and hence find K, the coefficient of thermal conductivity.

(e) What is the temperature profile, $T(y)$, of the gas in steady state?

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1. *Hard core gas:* A gas obeys the equation of state $P(V - Nb) = Nk_BT$, and has a heat capacity C_V independent of temperature. (N is kept fixed in the following.)

(a) Find the Maxwell relation involving $\partial S/\partial V|_{T,N}$.

(b) By calculating $dE(T, V)$, show that E is a function of T (and N) only.

(c) Show that $\gamma \equiv C_P/C_V = 1 + Nk_B/C_V$ (independent of T and V).

(d) By writing an expression for $E(P, V)$, or otherwise, show that an adiabatic change satisfies the equation $P(V - Nb)^\gamma$ =constant.

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2. *Energy of a gas:* The probability density to find a particle of momentum $\mathbf{p} \equiv (p_x, p_y, p_z)$ in a gas at temperature T is given by

$$p(\mathbf{p}) = \frac{1}{(2\pi m k_B T)^{3/2}} \exp\left(-\frac{p^2}{2mk_BT}\right), \qquad \text{where} \qquad p^2 = \mathbf{p}\cdot\mathbf{p} \quad .$$

(a) Using Wick's theorem, or otherwise, calculate the averages $\langle p^2 \rangle$ and $\langle(\mathbf{p}\cdot\mathbf{p})(\mathbf{p}\cdot\mathbf{p})\rangle$.

(b) Calculate the characteristic function for the energy $\varepsilon = p^2/2m$ of a gas particle.

(c) Using the characteristic function, or otherwise, calculate the m^{th} cumulant of the particle energy $\langle \varepsilon^m \rangle_c$.

(d) The total energy of a gas of N (independent) particles is given by $E = \sum_{i=1}^N \varepsilon_i$, where ε_i is the kinetic energy of the i^{th} particle, as given above. Use the central limit theorem to compute the probability density for energy, $p(E)$, for $N \gg 1$.

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3. *'Relativistic' gas:* Consider a gas of particles with a 'relativistic' one particle Hamiltonian $\mathcal{H}_1 = c|\mathbf{p}|$, where $|\mathbf{p}| = \sqrt{p_x^2 + p_y^2 + p_z^2}$ is the magnitude of the momentum. (The external potential is assumed to be zero, expect at the edges of the box confining the gas particles.) Throughout this problem treat the two body interactions and collisions precisely as in the case of classical particles considered in lectures.

(a) Write down the Boltzmann equation for the one-particle density $f_1(\mathbf{p}, \mathbf{q}, t)$, using the same collision form as employed in lectures (without derivation).

(b) The two body collisions conserve the number of particles, the momentum, and the particle energies as given by $\mathcal{H}_1$. Write down the most general form $f_1^0(\mathbf{p}, \mathbf{q}, t)$ that sets the collision integrand in the Boltzmann equation to zero. (You do not need to normalize this solution.)

For any function $\chi(\mathbf{p})$ which is conserved in the collisions, there is a hydrodynamic equation of the form

$$\partial_t \left(n \left\langle \chi \right\rangle\right) + \partial_\alpha \left(n \left\langle c\frac{p_\alpha}{|\mathbf{p}|}\chi\right\rangle\right) - n \left\langle \partial_t \chi \right\rangle - n \left\langle c\frac{p_\alpha}{|\mathbf{p}|}\partial_\alpha \chi\right\rangle = 0,$$

where $n(\mathbf{q}, t) = \int d^3\mathbf{p} f_1(\mathbf{p}, \mathbf{q}, t)$ is the local density, and

$$\langle \mathcal{O} \rangle = \frac{1}{n}\int d^3\mathbf{p} f_1(\mathbf{p}, \mathbf{q}, t)\mathcal{O}.$$

(c) Obtain the equation governing the density $n(\mathbf{q}, t)$, in terms of the average local velocity $u_\alpha = \langle cp_\alpha/|\mathbf{p}|\rangle$.

(d) Find the hydrodynamic equation for the local momentum density $\pi_\alpha(\mathbf{q}, t) \equiv \langle p_\alpha \rangle$, in terms of the pressure tensor $P_{\alpha\beta} = nc \left\langle \left(p_\alpha - \pi_\alpha\right)\left(p_\beta - \pi_\beta\right)/|\mathbf{p}|\right\rangle$.

(e) Find the (normalized) one particle density $f_1(\mathbf{p}, \mathbf{q}, t)$ for a gas of N such particles in a box of volume V, in equilibrium at a temperature T.

(f) Evaluate the pressure tensor $P_{\alpha\beta}$ for the above gas in equilibrium at temperature T.

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1. *Wire:* Experiments on stretching an elastic wire indicate that, at a temperature T, a displacement x requires a force

$$J = ax - bT + cTx,$$

where a, b, and c are constants. Furthermore, its heat capacity at constant displacement is proportional to temperature, i.e. $C_x = A(x)T$.

(a) Use an appropriate Maxwell relation to calculate $\partial S/\partial x|_T$.

(b) Show that A has to be independent of x, i.e. $dA/dx = 0$.

(c) Give the expression for $S(T, x)$, and comment on whether it is compatible with the third law of thermodynamics.

(d) Calculate the heat capacity at constant tension, i.e. $C_J = T\, \partial S/\partial T|_J$, as a function of T and J.

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2. *Random matrices:* As a model for energy levels of complex nuclei, Wigner considered $N \times N$ symmetric matrices whose elements are random. Let us assume that each element M_{ij} (for $i \geq j$) is an independent random variable taken from the probability density function

$$p(M_{ij}) = \frac{1}{2a} \quad \text{for} \quad -a < M_{ij} < a\,, \quad \text{and} \quad p(M_{ij}) = 0 \quad \text{otherwise.}$$

(a) Calculate the characteristic function for each element M_{ij}.

(b) Calculate the characteristic function for the trace of the matrix, $T \equiv \mathrm{tr}\, M = \sum_i M_{ii}$.

(c) What does the central limit theorem imply about the probability density function of the trace at large N?

(d) For large N, each eigenvalue λ_α $(\alpha = 1, 2, \cdots, N)$ of the matrix M is distributed according to a probability density function

$$p(\lambda) = \frac{2}{\pi\lambda_0}\sqrt{1 - \frac{\lambda^2}{\lambda_0^2}} \quad \text{for} \quad -\lambda_0 < \lambda < \lambda_0, \quad \text{and} \quad p(\lambda) = 0 \quad \text{otherwise,}$$

(known as the Wigner semi-circle rule). Find the variance of λ.
(**Hint:** Changing variables to $\lambda = \lambda_0 \sin\theta$ simplifies the integrals.)

(e) If in the previous result, we have $\lambda_0^2 = 4Na^2/3$, can the eigenvalues be independent of each other?

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3. *Viscosity:* Consider a classical gas between two plates separated by a distance w. One plate at $y = 0$ is stationary, while the other at $y = w$ moves with a constant velocity $v_x = u$. A zeroth order approximation to the one particle density is,

$$f_1^0(\vec{p}, \vec{q}) = \frac{n}{(2\pi m k_B T)^{3/2}} \exp\left[-\frac{1}{2mk_BT}\left((p_x - m\alpha y)^2 + p_y^2 + p_z^2\right)\right],$$

obtained from the *uniform* Maxwell–Boltzmann distribution by substituting the average value of the gas velocity at each point. ($\alpha = u/w$ is the velocity gradient, while n and T are constants.)

(a) The above approximation does not satisfy the Boltzmann equation as the collision term (right hand side of the equation) vanishes, while (the left hand side) $df_1^0/dt \neq 0$. Find a better approximation, $f_1^1(\vec{p})$, by considering the linearized Boltzmann equation in the single collision time approximation, i.e.

$$\mathcal{L}\left[f_1^1\right] \approx \left[\frac{\partial}{\partial t} + \frac{\vec{p}}{m} \cdot \frac{\partial}{\partial \vec{q}}\right] f_1^0 \approx -\frac{f_1^1 - f_1^0}{\tau_\times},$$

where $\tau_\times$ is a characteristic mean time between collisions.

(b) Calculate the off-diagonal component $P_{xy}(y)$ of the pressure tensor.

(c) The gas exerts a transverse force per unit area $F_x = -P_{xy}(y = w)$ on the moving plate. Calculate this force, and hence obtain the coefficient of viscosity, defined by $\eta = F_x/\alpha$.

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