According to MIT regulations, no problem set can have a due date later than 12/9/05. However, you can be (and will be!) examined on material that is covered in December. Consequently, this problem set is designed to help you learn the new material, but has no due date. The solutions will be available on **12/14/05**.

1. *Numerical estimates:*

(a) Compare the thermal wavelength of a neutron at room temperature to the minimum wavelength of a phonon in a typical crystal.

(b) Estimate the degeneracy discriminant, $n\lambda^3$, for hydrogen, helium, and oxygen gases at room temperature and pressure. At what temperatures do quantum mechanical effects become important for these gases?

(c) Experiments on He^4 indicate that at temperatures below $1^o K$, the heat capacity is given by $C_V = 20.4 T^3 J K g^{-1}\, {}^o K^{-1}$. Find the low energy excitation spectrum, $\mathcal{E}(k)$, of He^4. (Hint: There is only one non-degenerate branch of such excitations.)

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2. He^3 at low temperatures can be converted from liquid to solid by application of pressure. A peculiar feature of its phase boundary is that $(dP/dT)_{\text{melting}}$ is negative at temperatures below $0.3\,{}^o K$ $[(dP/dT)_m \approx -30\text{atm}\ {}^o K^{-1}$ at $T \approx 0.1\,{}^o K]$. We will use a simple model of liquid and solid phases of He^3 to account for this feature.

(a) In the solid phase, the He^3 atoms form a crystal lattice. Each atom has nuclear spin of 1/2. Ignoring the interaction between spins, what is the entropy per particle s_s, due to the spin degrees of freedom?

(b) Liquid He^3 is modelled as an ideal Fermi gas, with a volume of $46 \mathring{A}^3$ per atom. What is its Fermi temperature T_F, in degrees Kelvin?

(c) How does the heat capacity of liquid He^3 behave at low temperatures? Write down an expression for C_V in terms of N, T, k_B, T_F, *up to a numerical constant*, that is valid for $T \ll T_F$.

(d) Using the result in (c), calculate the entropy per particle s_ℓ, in the liquid at low temperatures. For $T \ll T_F$, which phase (solid or liquid) has the higher entropy?

(e) By equating chemical potentials, or by any other technique, prove the Clausius–Clapeyron equation $(dP/dT)_{\text{melting}} = (s_\ell - s_s)/(v_\ell - v_s)$, where v_ℓ and v_s are the volumes per particle in the liquid and solid phases respectively.

(f) It is found experimentally that $v_\ell - v_s = 3\mathring{A}^3$ per atom. Using this information, plus the results in parts (a), (d), and (e), estimate $(dP/dT)_{\rm melting}$ at $T \ll T_F$.

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3. *Boson Ferromagnetism:* Consider a gas of non-interacting spin 1 bosons, each subject to a Hamiltonian

$$\mathcal{H}_1(\vec{p}, s_z) = \frac{\vec{p}^{\,2}}{2m} - \mu_0 s_z B \quad ,$$

where $\mu_0 = e\hbar/mc$, and s_z takes *three* possible values of (-1, 0, +1). (The orbital effect, $\vec{p} \to \vec{p} - e\vec{A}$, has been ignored.)

(a) In a grand canonical ensemble of chemical potential μ, what are the average occupation numbers $\left\{\langle n_+(\vec{k})\rangle, \langle n_0(\vec{k})\rangle, \langle n_-(\vec{k})\rangle\right\}$, of one-particle states of wavenumber $\vec{k} = \vec{p}/\hbar$?

(b) Calculate the average total numbers $\{N_+, N_0, N_-\}$, of bosons with the three possible values of s_z in terms of the functions $f_m^+(z)$.

(c) Write down the expression for the magnetization $M(T, \mu) = \mu_0(N_+ - N_-)$, and by expanding the result for small B find the *zero field susceptibility* $\chi(T, \mu) = \partial M/\partial B|_{B=0}$.

To find the behavior of $\chi(T, n)$, where $n = N/V$ is the total density, proceed as follows:

(d) For $B = 0$, find the high temperature expansion for $z(\beta, n) = e^{\beta\mu}$, correct to second order in n. Hence obtain the first correction from quantum statistics to $\chi(T, n)$ at high temperatures.

(e) Find the temperature $T_c(n, B = 0)$, of Bose-Einstein condensation. What happens to $\chi(T, n)$ on approaching $T_c(n)$ from the high temperature side?

(f) What is the chemical potential μ for $T < T_c(n)$, at a small but finite value of B? Which one-particle state has a macroscopic occupation number?

(g) Using the result in (f), find the spontaneous magnetization,

$$\overline{M}(T, n) = \lim_{B \to 0} M(T, n, B).$$

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Suggested Reading: Huang, Chapters 12 and 13.