Review Problems

The Mid-term quiz will take place on Monday **10/24/05** in room **1-190** from **2:30 to 4:00** pm. There will be a recitation with quiz review on Friday **10/21/05**.

All topics up to (but not including) the micro-canonical ensemble will be covered. The exam is 'closed book,' but if you wish you may bring a two-sided sheet of formulas. The enclosed exams (and solutions) from the previous years are intended to help you review the material. *Solutions to the midterm are available in the exams section.*

<p align="center">********</p>

Answer all three problems, but note that the first parts of each problem are easier than its last parts. Therefore, make sure to proceed to the next problem when you get stuck.

You may find the following information helpful:

Physical Constants

Electron mass	$m_e \approx 9.1 \times 10^{-31} Kg$	Proton mass	$m_p \approx 1.7 \times 10^{-27} Kg$
Electron Charge	$e \approx 1.6 \times 10^{-19} C$	Planck's const./2π	$\hbar \approx 1.1 \times 10^{-34} Js^{-1}$
Speed of light	$c \approx 3.0 \times 10^{8} ms^{-1}$	Stefan's const.	$\sigma \approx 5.7 \times 10^{-8} Wm^{-2}K^{-4}$
Boltzmann's const.	$k_B \approx 1.4 \times 10^{-23} JK^{-1}$	Avogadro's number	$N_0 \approx 6.0 \times 10^{23} mol^{-1}$

Conversion Factors

$$1atm \equiv 1.0 \times 10^5 Nm^{-2} \qquad 1\text{Å} \equiv 10^{-10}m \qquad 1eV \equiv 1.1 \times 10^4 K$$

Thermodynamics

$$dE = TdS + \bar{d}W \qquad \text{For a gas: } \bar{d}W = -PdV \qquad \text{For a wire: } \bar{d}W = Jdx$$

Mathematical Formulas

$$\int_0^\infty dx\, x^n\, e^{-\alpha x} = \frac{n!}{\alpha^{n+1}} \qquad\qquad \left(\tfrac{1}{2}\right)! = \frac{\sqrt{\pi}}{2}$$

$$\int_{-\infty}^\infty dx \exp\left[-ikx - \frac{x^2}{2\sigma^2}\right] = \sqrt{2\pi\sigma^2} \exp\left[-\frac{\sigma^2 k^2}{2}\right] \qquad \lim_{N\to\infty} \ln N! = N\ln N - N$$

$$\left\langle e^{-ikx}\right\rangle = \sum_{n=0}^{\infty} \frac{(-ik)^n}{n!} \left\langle x^n \right\rangle \qquad\qquad \ln\left\langle e^{-ikx}\right\rangle = \sum_{n=1}^{\infty} \frac{(-ik)^n}{n!} \left\langle x^n \right\rangle_c$$

$$\cosh(x) = 1 + \frac{x^2}{2!} + \frac{x^4}{4!} + \cdots \qquad\qquad \sinh(x) = x + \frac{x^3}{3!} + \frac{x^5}{5!} + \cdots$$

Surface area of a unit sphere in d dimensions $\qquad S_d = \frac{2\pi^{d/2}}{(d/2-1)!}$

1. *Photon gas Carnot cycle:* The aim of this problem is to obtain the blackbody radiation relation, $E(T, V) \propto VT^4$, starting from the equation of state, by performing an infinitesimal Carnot cycle on the photon gas.
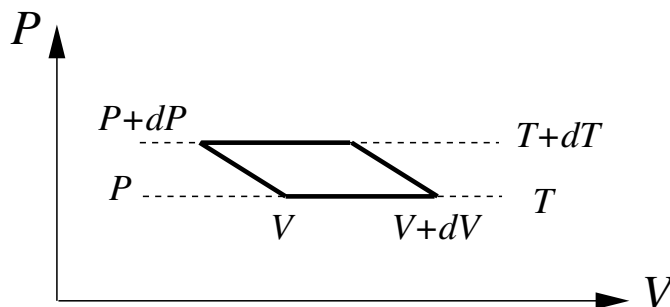
(a) Express the work done, W, in the above cycle, in terms of dV and dP.

● Ignoring higher order terms, net work is the area of the cycle, given by $W = dPdV$.

(b) Express the heat absorbed, Q, in expanding the gas *along an isotherm*, in terms of P, dV, and an appropriate derivative of $E(T, V)$.

● Applying the first law, the heat absorbed is

$$Q = dE + PdV = \left[\left(\frac{\partial E}{\partial T} \right)_{V\square} dT + \left(\frac{\partial E}{\partial V} \right)_T dV + PdV \right]_{\text{isotherm}} = \left[\left(\frac{\partial E}{\partial V} \right)_T + P \right] dV.$$

(c) Using the efficiency of the Carnot cycle, relate the above expressions for W and Q to T and dT.

● The efficiency of the Carnot cycle ($\eta = dT/T$) is here calculated as

$$\eta = \frac{W}{Q} = \square \frac{dP}{[(\partial E/\partial V)_T + P]} = \frac{dT}{T}.$$

(d) Observations indicate that the pressure of the photon gas is given□by $P = AT^4$, where $A = \pi^2 k_B^4 / 45 \, (\hbar c)^3$ is a constant. Use this information to obtain $E(T, V)$, assuming $E(0, V) = 0$.

● From the result of part (c) and the relation $P = AT^4$,

$$4AT^4 = \left(\frac{\partial E}{\partial V\square} \right)_{T\square}^{\square} + AT^4, \quad \text{or} \quad \left(\frac{\partial E}{\partial V} \right)_T = 3AT^4,$$

so that

$$E = 3AVT^4.$$

(e) Find the relation describing the *adiabatic paths* in the above cycle.

• Adiabatic curves are given by $dQ = 0$, or

$$0 = \left(\frac{\partial E}{\partial T}\right)_V dT + \left(\frac{\partial E}{\partial V}\right)_T dV + PdV = 3VdP + 4PdV,$$

i.e.

$$PV^{4/3} = \text{constant.}$$

$$********$$

2. *Moments of momentum:* Consider a gas of N classical particles of mass m in thermal equilibrium at a temperature T, in a box of volume V.

(a) Write down the equilibrium one particle density $f_{\text{eq.}}(\vec{p}, \vec{q})$, for coordinate $\vec{q}$, and momentum $\vec{p}$.

• The equilibrium Maxwell-Boltzmann distribution reads

$$f(\vec{p}, \vec{q}) = \frac{n}{(2\pi m k_B T)^{3/2}} \exp\left(-\frac{p^2}{2m k_B T}\right).$$

(b) Calculate the joint characteristic function, $\left\langle \exp\left(-i\vec{k} \cdot \vec{p}\right) \right\rangle$, for momentum.

• Performing the Gaussian average yields

$$\tilde{p}\left(\vec{k}\right) = \left\langle e^{-i\vec{k} \cdot \vec{p}} \right\rangle = \exp\left(-\frac{m k_B T}{2} k^2\right).$$

(c) Find all the joint cumulants $\left\langle p_x^\ell p_y^m p_z^n \right\rangle_c$.

• The cumulants are calculated from the characteristic function, as

$$\left\langle p_x^\ell p_y^m p_z^n \right\rangle_c = \left[\frac{\partial}{\partial(-ik_x)}\right]^\ell \left[\frac{\partial}{\partial(-ik_y)}\right]^m \left[\frac{\partial}{\partial(-ik_z)}\right]^n \ln \tilde{p}\left(\vec{k}\right)\Bigg|_{\vec{k}=0}$$

$$= m k_B T \left(\delta_{\ell 2}\delta_{m0}\delta_{n0} + \delta_{\ell 0}\delta_{m2}\delta_{n0} + \delta_{\ell 0}\delta_{m0}\delta_{n2}\right),$$

i.e., there are only second cumulants; all other cumulants are zero.

(d) Calculate the joint moment $\left\langle p_\alpha p_\beta \left(\vec{p} \cdot \vec{p}\right)\right\rangle$.

• Using Wick's theorem

$$\left\langle p_\alpha p_\beta \left(\vec{p} \cdot \vec{p}\right)\right\rangle = \left\langle p_\alpha p_\beta p_\gamma p_\gamma \right\rangle$$

$$= \left\langle p_\alpha p_\beta \right\rangle \left\langle p_\gamma p_\gamma \right\rangle + 2 \left\langle p_\alpha p_\gamma \right\rangle \left\langle p_\beta p_\gamma \right\rangle$$

$$= (m k_B T)^2 \delta_{\alpha\beta}\delta_{\gamma\gamma} + 2 (m k_B T)^2 \delta_{\alpha\gamma}\delta_{\beta\gamma}$$

$$= 5 (m k_B T)^2 \delta_{\alpha\beta}.$$

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Alternatively, directly from the characteristic function,

$$
\begin{aligned}
\langle p_\alpha p_\beta \left(\vec{p} \cdot \vec{p} \right) \rangle &= \left. \frac{\partial}{\partial \left(-ik_\alpha \right)} \frac{\partial}{\partial \left(-ik_\beta \right)} \frac{\partial}{\partial \left(-ik_\gamma \right)} \frac{\partial}{\partial \left(-ik_\gamma \right)} \tilde{p}\left(\vec{k} \right) \right|_{\vec{k}=0} \\
&= \left. \frac{\partial}{\partial \left(-ik_\alpha \right)} \frac{\partial}{\partial \left(-ik_\beta \right)} \left[3mk_B - (mk_BT)^2 \, \vec{k}^2 \right] e^{-\frac{mk_BT}{2} \vec{k}^2} \right|_{\vec{k}=0} \\
&= 5 \left(mk_BT \right)^2 \delta_{\alpha\beta}.
\end{aligned}
$$

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3. *Light and matter:* In this problem we use kinetic theory to explore the equilibrium between atoms and radiation.

(a) The atoms are assumed to be either in their ground state a_0, or in an excited state a_1, which has a higher energy ε. By considering the atoms as a collection of N fixed two-state systems of energy E (i.e. ignoring their coordinates and momenta), calculate the ratio n_1/n_0 of densities of atoms in the two states as a function of temperature T.

• The energy and temperature of a two-state system are related by

$$
E = \frac{N\epsilon}{1 + \exp\left(\epsilon/k_BT\right)},
$$

leading to

$$
n_0 = \frac{N - E/\epsilon}{V} = \frac{N}{V} \frac{\exp\left(\epsilon/k_BT\right)}{1 + \exp\left(\epsilon/k_BT\right)}, \quad \text{and} \quad n_1 = \frac{E/\epsilon}{V} = \frac{N}{V} \frac{1}{1 + \exp\left(\epsilon/k_BT\right)},
$$

so that

$$
\frac{n_1}{n_0} = \exp\left(-\frac{\epsilon}{k_BT} \right).
$$

Consider photons γ of frequency $\omega = \varepsilon/\hbar$ and momentum $|\vec{p}| = \hbar\omega/c$, which can interact with the atoms through the following processes:

(i) *Spontaneous emission:* $a_1 \to a_0 + \gamma$.

(ii) *Adsorption:* $a_0 + \gamma \to a_1$.

(iii) *Stimulated emission:* $a_1 + \gamma \to a_0 + \gamma + \gamma$.

Assume that spontaneous emission occurs with a probability σ_{sp}, and that adsorption and stimulated emission have constant (angle-independent) differential cross-sections of $\sigma_{\text{ad}}/4\pi$ and $\sigma_{\text{st}}/4\pi$, respectively.

(b) Write down the Boltzmann equation governing the density f of the photon gas, treating the atoms as fixed scatterers of densities n_0 and n_1.

• The Boltzmann equation for photons in the presence of fixed scatterers reads

$$
\frac{\partial f}{\partial t} + \vec{p} \cdot \frac{\partial f}{\partial \vec{q}} = -\sigma_{\text{ad}} n_0 cf + \sigma_{\text{st}} n_1 cf + \sigma_{\text{sp}} n_1.
$$

(c) Find the equilibrium density $f_{\text{eq.}}$ for the photons of the above frequency.

• In uniform equilibrium, the left-hand side vanishes, leaving

$$-\sigma_{\text{ad}} n_0 c f_{\text{eq.}} + \sigma_{\text{st}} n_1 c f_{\text{eq.}} + \sigma_{\text{sp}} n_1 = 0,$$

i.e.

$$f_{\text{eq.}} = \frac{1}{c} \frac{\sigma_{\text{sp}}}{\sigma_{\text{ad}} n_0 / n_1 - \sigma_{\text{st}}} = \frac{1}{c} \frac{\sigma_{\text{sp}}}{\sigma_{\text{ad}} \exp\left(\epsilon/k_B T\right) - \sigma_{\text{st}}}.$$

(d) According to Planck's law, the density of photons at a temperature T depends on their frequency ω as $f_{\text{eq.}} = \left[\exp\left(\hbar\omega/k_B T\right) - 1\right]^{-1}/h^3$. What does this imply about the above cross sections?

• The result of part (c) agrees with Planck's law if

$$\sigma_{\text{ad}} = \sigma_{\text{st}}, \quad \text{and} \quad \sigma_{\text{sp}} = \frac{c}{h^3} \sigma_{\text{st}},$$

a conclusion first reached by Einstein, and verified later with explicit quantum mechanical calculations of cross-sections.

(e) Consider a situation in which light shines along the x axis on a collection of atoms whose boundary coincides with the $x = 0$ plane, as illustrated in the figure.
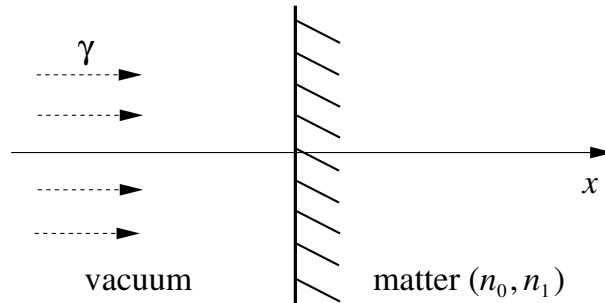
Clearly, f will depend on x (and p_x), but will be independent of y and z. Adapt the Boltzmann equation you propose in part (b) to the case of a uniform incoming flux of photons with momentum $\vec{p} = \hbar\omega\hat{x}/c$. What is the *penetration length* across which the incoming flux decays?

• In this situation, the Boltzmann equation reduces to

$$p_x \frac{\partial f}{\partial x} = \sigma_{\text{st}} c \left[\left(n_1 - n_0\right) f + \frac{n_1}{h^3}\right] \theta\left(x\right).$$

To the uniform solution obtained before, one can add an exponentially decaying term for $x > 0$, i.e.

$$f\left(p_x, x > 0\right) = A\left(p_x\right) e^{-ax/p_x} + f_{\text{eq.}}\left(p_x\right).$$

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The constant $A(p_x)$ can be determined by matching to solution for $x < 0$ at $x = 0$, and is related to the incoming flux. The penetration depth d is the inverse of the decay parameter, and given by

$$d = \frac{p_x}{a}, \qquad \text{with} \qquad a = \sigma_{\text{st}} c (n_0 - n_1) > 0.$$

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1. *Superconducting transition:* Many metals become superconductors at low temperatures T, and magnetic fields B. The heat capacities of the two phases at zero magnetic field are approximately given by

$$\begin{cases} C_s(T) = V\alpha T^3 & \text{in the superconducting phase} \\ C_n(T) = V\left[\beta T^3 + \gamma T\right] & \text{in the normal phase} \end{cases},$$

where V is the volume, and $\{\alpha, \beta, \gamma\}$ are constants. (There is no appreciable change in volume at this transition, and mechanical work can be ignored throughout this problem.)

(a) Calculate the entropies $S_s(T)$ and $S_n(T)$ of the two phases at zero field, using the third law of thermodynamics.

● Finite temperature entropies are obtained by integrating $dS = đQ/T$, starting from $S(T=0) = 0$. Using the heat capacities to obtain the heat inputs, we find

$$\begin{cases} C_s = V\alpha T^3 = T\dfrac{dS_s}{dT}, & \Longrightarrow \quad S_s = V\dfrac{\alpha T^3}{3}, \\ C_n = V\left[\beta T^3 + \gamma T\right] = T\dfrac{dS_n}{dT}, & \Longrightarrow \quad S_n = V\left[\dfrac{\beta T^3}{3} + \gamma T\right] \end{cases}.$$

(b) Experiments indicate that there is no latent heat ($L = 0$) for the transition between the normal and superconducting phases at zero field. Use this information to obtain the transition temperature T_c, as a function of α, β, and γ.

● The Latent hear for the transition is related to the difference in entropies, and thus

$$L = T_c\left(S_n(T_c) - S_s(T_c)\right) = 0.$$

Using the entropies calculated in the previous part, we obtain

$$\frac{\alpha T_c^3}{3} = \frac{\beta T_c^3}{3} + \gamma T_c, \quad \Longrightarrow \quad T_c = \sqrt{\frac{3\gamma}{\alpha - \beta}}.$$

(c) At zero temperature, the electrons in the superconductor form bound Cooper pairs. As a result, the internal energy of the superconductor is reduced by an amount $V\Delta$, i.e. $E_n(T=0) = E_0$ and $E_s(T=0) = E_0 - V\Delta$ for the metal and superconductor, respectively. Calculate the internal energies of both phases at finite temperatures.

● Since $dE = TdS + BdM + \mu dN$, for $dN = 0$, and $B = 0$, we have $dE = TdS = CdT$. Integrating the given expressions for heat capacity, and starting with the internal energies E_0 and $E_0 - V\Delta$ at $T = 0$, yields

$$\begin{cases} E_s(T) = E_0 + V\left[-\Delta + \dfrac{\alpha}{4}T^4\right] \\ E_n(T) = E_0 + V\left[\dfrac{\beta}{4}T^4 + \dfrac{\gamma}{2}T^2\right] \end{cases}.$$

(d) By comparing the Gibbs free energies (or chemical potentials) in the two phases, obtain an expression for the energy gap Δ in terms of α, β, and γ.

• The Gibbs free energy $G = E - TS - BM = \mu N$ can be calculated for $B = 0$ in each phase, using the results obtained before, as

$$
\begin{cases}
G_s(T) = E_0 + V\left[-\Delta + \dfrac{\alpha}{4}T^4\right] - TV\dfrac{\alpha}{3}T^3 = E_0 - V\left[\Delta + \dfrac{\alpha}{12}T^4\right] \\[2mm]
G_n(T) = E_0 + V\left[\dfrac{\beta}{4}T^4 + \dfrac{\gamma}{2}T^2\right] - TV\left[\dfrac{\beta}{3}T^3 + \gamma T\right] = E_0 - V\left[\dfrac{\beta}{12}T^4 + \dfrac{\gamma}{2}T^2\right]
\end{cases}.
$$

At the transition point, the chemical potentials (and hence the Gibbs free energies) must be equal, leading to

$$
\Delta + \frac{\alpha}{12}T_c^4 = \frac{\beta}{12}T_c^4 + \frac{\gamma}{2}T_c^2, \quad \Longrightarrow \quad \Delta = \frac{\gamma}{2}T_c^2 - \frac{\alpha - \beta}{12}T_c^4.
$$

Using the value of $T_c = \sqrt{3\gamma/(\alpha - \beta)}$, we obtain

$$
\Delta = \frac{3}{4}\frac{\gamma^2}{\alpha - \beta}.
$$

(e) In the presence of a magnetic field B, inclusion of magnetic work results in $dE = TdS + BdM + \mu dN$, where M is the magnetization. The superconducting phase is a perfect diamagnet, expelling the magnetic field from its interior, such that $M_s = -VB/(4\pi)$ in appropriate units. The normal metal can be regarded as approximately non-magnetic, with $M_n = 0$. Use this information, in conjunction with previous results, to show that the superconducting phase becomes normal for magnetic fields larger than

$$
B_c(T) = B_0\left(1 - \frac{T^2}{T_c^2}\right),
$$

giving an expression for B_0.

• Since $dG = -SdT - MdB + \mu dN$, we have to add the integral of $-MdB$ to the Gibbs free energies calculated in the previous section for $B = 0$. There is no change in the metallic phase since $M_n = 0$, while in the superconducting phase there is an additional contribution of $-\int M_s dB = (V/4\pi)\int B dB = (V/8\pi)B^2$. Hence the Gibbs free energies at finite field are

$$
\begin{cases}
G_s(T, B) = E_0 - V\left[\Delta + \dfrac{\alpha}{12}T^4\right] + V\dfrac{B^2}{8\pi} \\[2mm]
G_n(T, B) = E_0 - V\left[\dfrac{\beta}{12}T^4 + \dfrac{\gamma}{2}T^2\right]
\end{cases}.
$$

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Equating the Gibbs free energies gives a critical magnetic field

$$\frac{B_c^2}{8\pi} = \Delta - \frac{\gamma}{2}T^2 + \frac{\alpha - \beta}{12}T^4 = \frac{3}{4}\frac{\gamma^2}{\alpha - \beta} - \frac{\gamma}{2}T^2 + \frac{\alpha - \beta}{12}T^4$$

$$= \frac{\alpha - \beta}{12}\left[\left(\frac{3\gamma}{\alpha - \beta}\right)^2 - \frac{6\gamma T^2}{\alpha - \beta} + T^4\right] = \frac{\alpha - \beta}{12}\left(T_c^2 - T^2\right)^2,$$

where we have used the values of Δ and T_c obtained before. Taking the square root of the above expression gives

$$B_c = B_0\left(1 - \frac{T^2}{T_c^2}\right), \quad \text{where} \quad B_0 = \sqrt{\frac{2\pi(\alpha - \beta)}{3}}T_c^2 = \sqrt{\frac{6\pi\gamma^2}{\alpha - \beta}} = T_c\sqrt{2\pi\gamma}.$$

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2. *Probabilities:* Particles of type A or B are chosen independently with probabilities p_A and p_B.

(a) What is the probability $p(N_A, N)$ that N_A out of the N particles are of type A?
- The answer is the binomial probability distribution

$$p(N_A, N) = \frac{N!}{N_A!(N - N_A)!}p_A^{N_A}p_B^{N - N_B}.$$

(b) Calculate the mean and the variance of N_A.
- We can write

$$n_A = \sum_{i=1}^{N} t_i,$$

where $t_i = 1$ if the i-th particle is A, and 0 if it is B. The mean value is then equal to

$$\langle N_A \rangle = \sum_{i=1}^{N} \langle t_i \rangle = \sum_{i=1}^{N}(p_A \times 1 + p_B \times 0) = N p_A.$$

Similarly, since the $\{t_i\}$ are independent variables,

$$\langle N_A^2 \rangle_c = \sum_{i=1}^{N}\left(\langle t_i^2 \rangle - \langle t_i \rangle^2\right) = \sum_{i=1}^{N}\left(p_A - p_A^2\right) = N p_A p_B.$$

(c) Use the central limit theorem to obtain the probability $p(N_A, N)$ for large N.

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• According to the *central limit theorem* the PDF of the sum of independent variables for large N approaches a Gaussian of the right mean and variance. Using the mean and variance calculated in the previous part, we get

$$\lim_{N \gg 1} p(N_A, N) \approx \exp\left[-\frac{(N_A - Np_A)^2}{2Np_Ap_B}\right] \frac{1}{\sqrt{2\pi Np_Ap_B}}.$$

(d) Apply Stirling's approximation ($\ln N! \approx N \ln N - N$) to $\ln p(N_A, N)$ [using the probability calculated in part (a), **not** part (c)] to find the most likely value, $\overline{N_A}$, for $N \gg 1$.

• Applying Stirling's approximation to the logarithm of the binomial distribution gives

$$\ln p(N_A, N) = \ln N! - \ln N_A! - \ln(N - N_A)! + N_A \ln p_A + (N - N_A) \ln p_B$$
$$\approx -N_A \ln\left(\frac{N_A}{N}\right) - (N - N_A) \ln\left(1 - \frac{N_A}{N}\right) + N_A \ln p_A + (N - N_A) \ln p_B.$$

The most likely value, $\overline{N_A}$, is obtained by setting the derivative of the above expression with respect to N_A to zero, i.e.

$$\frac{d \ln p}{dN_A} = -\ln\left[\frac{\overline{N_A}}{N}\frac{N}{N - \overline{N_A}}\right] + \ln\frac{p_A}{p_B} = 0, \implies \overline{N_A} = p_A N.$$

Thus the most likely value is the same as the mean in this limit.

(e) Expand $\ln p(N_A, N)$ calculated in (d) around its maximum to second order in $\left(N_A - \overline{N_A}\right)$, and check for consistency with the result from the central limit theorem.

• Taking a second derivative of $\ln p$ gives

$$\frac{d^2 \ln p}{dN_A^2} = -\frac{1}{\overline{N_A}} - \frac{1}{N - \overline{N_A}} = -\frac{N}{\overline{N_A}\left(N - \overline{N_A}\right)} = -\frac{1}{Np_Ap_B}.$$

The expansion of $\ln p$ around its maximum thus gives

$$\ln p \approx -\frac{(N_A - p_A N)^2}{2Np_Ap_B},$$

which is consistent with the result from the central limit theorem. The correct normalization is also obtained if the next term in the Stirling approximation is included.

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3. *Thermal Conductivity:* Consider a classical gas between two plates separated by a distance w. One plate at $y = 0$ is maintained at a temperature T_1, while the other plate at

$y = w$ is at a different temperature T_2. The gas velocity is zero, so that the initial zeroth order approximation to the one particle density is,

$$f_1^0(\vec{p}, x, y, z) = \frac{n(y)}{[2\pi m k_B T(y)]^{3/2}} \exp\left[-\frac{\vec{p} \cdot \vec{p}}{2m k_B T(y)}\right].$$

(a) What is the necessary relation between $n(y)$ and $T(y)$, to ensure that the gas velocity $\vec{u}$ remains zero? (Use this relation between $n(y)$ and $T(y)$ in the remainder of this problem.)

• Since there is no external force acting on the gas between plates, the gas can only flow locally if there are variations in pressure. Since the local pressure is $P(y) = n(y)k_B T(y)$, the condition for the fluid to be stationary is

$$n(y)T(y) = \text{constant}.$$

(b) Using Wick's theorem, or otherwise, show that

$$\left\langle p^2 \right\rangle^0 \equiv \left\langle p_\alpha p_\alpha \right\rangle^0 = 3\,(mk_B T), \quad \text{and} \quad \left\langle p^4 \right\rangle^0 \equiv \left\langle p_\alpha p_\alpha p_\beta p_\beta \right\rangle^0 = 15\,(mk_B T)^2,$$

where $\langle \mathcal{O} \rangle^0$ indicates local averages with the Gaussian weight f_1^0. Use the result $\left\langle p^6 \right\rangle^0 = 105(mk_B T)^3$ (*you don't have to derive this*) in conjunction with symmetry arguments to conclude

$$\left\langle p_y^2 p^4 \right\rangle^0 = 35\,(mk_B T)^3.$$

• The Gaussian weight has a covariance $\left\langle p_\alpha p_\beta \right\rangle^0 = \delta_{\alpha\beta}(mk_B T)$. Using Wick's theorem gives

$$\left\langle p^2 \right\rangle^0 = \left\langle p_\alpha p_\alpha \right\rangle^0 = (mk_B T)\,\delta_{\alpha\alpha} = 3\,(mk_B T).$$

Similarly

$$\left\langle p^4 \right\rangle^0 = \left\langle p_\alpha p_\alpha p_\beta p_\beta \right\rangle^0 = (mk_B T)^2\,(\delta_{\alpha\alpha} + 2\delta_{\alpha\beta}\delta_{\alpha\beta}) = 15\,(mk_B T)^2.$$

The symmetry along the three directions implies

$$\left\langle p_x^2 p^4 \right\rangle^0 = \left\langle p_y^2 p^4 \right\rangle^0 = \left\langle p_z^2 p^4 \right\rangle^0 = \frac{1}{3}\left\langle p^2 p^4 \right\rangle^0 = \frac{1}{3} \times 105\,(mk_B T)^3 = 35\,(mk_B T)^3.$$

(c) The zeroth order approximation does not lead to relaxation of temperature/density variations related as in part (a). Find a better (time independent) approximation $f_1^1(\vec{p}, y)$, by linearizing the Boltzmann equation in the single collision time approximation, to

$$\mathcal{L}\left[f_1^1\right] \approx \left[\frac{\partial}{\partial t} + \frac{p_y}{m}\frac{\partial}{\partial y}\right] f_1^0 \approx -\frac{f_1^1 - f_1^0}{\tau_K},$$

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where τ_K is of the order of the mean time between collisions.

- Since there are only variations in y, we have

$$\left[\frac{\partial}{\partial t} + \frac{p_y}{m}\frac{\partial}{\partial y}\right]f_1^0 = f_1^0 \frac{p_y}{m}\partial_y \ln f_1^0 = f_1^0 \frac{p_y}{m}\partial_y \left[\ln n - \frac{3}{2}\ln T - \frac{p^2}{2mk_BT} - \frac{3}{2}\ln(2\pi mk_B)\right]$$

$$= f_1^0 \frac{p_y}{m}\left[\frac{\partial_y n}{n} - \frac{3}{2}\frac{\partial_y T}{T} + \frac{p^2}{2mk_BT}\frac{\partial T}{T}\right] = f_1^0 \frac{p_y}{m}\left[-\frac{5}{2} + \frac{p^2}{2mk_BT}\right]\frac{\partial_y T}{T},$$

where in the last equality we have used $nT = \text{constant}$ to get $\partial_y n/n = -\partial_y T/T$. Hence the first order result is

$$f_1^1(\vec{p}, y) = f_1^0(\vec{p}, y)\left[1 - \tau_K \frac{p_y}{m}\left(\frac{p^2}{2mk_BT} - \frac{5}{2}\right)\frac{\partial_y T}{T}\right].$$

(d) Use f_1^1, along with the averages obtained in part (b), to calculate h_y, the y component of the heat transfer vector, and hence find K, the coefficient of thermal conductivity.

- Since the velocity $\vec{u}$ is zero, the heat transfer vector is

$$h_y = n\left\langle c_y \frac{mc^2}{2}\right\rangle^1 = \frac{n}{2m^2}\langle p_y p^2\rangle^1.$$

In the zeroth order Gaussian weight all odd moments of p have zero average. The corrections in f_1^1, however, give a non-zero heat transfer

$$h_y = -\tau_K \frac{n}{2m^2}\frac{\partial_y T}{T}\left\langle \frac{p_y}{m}\left(\frac{p^2}{2mk_BT} - \frac{5}{2}\right)p_y p^2\right\rangle^0.$$

Note that we need the Gaussian averages of $\langle p_y^2 p^4\rangle^0$ and $\langle p_y^2 p^2\rangle^0$. From the results of part (b), these averages are equal to $35(mk_BT)^3$ and $5(mk_BT)^2$, respectively. Hence

$$h_y = -\tau_K \frac{n}{2m^3}\frac{\partial_y T}{T}(mk_BT)^2\left(\frac{35}{2} - \frac{5\times 5}{2}\right) = -\frac{5}{2}\frac{n\tau_K k_B^2 T}{m}\partial_y T.$$

The coefficient of thermal conductivity relates the heat transferred to the temperature gradient by $\vec{h} = -K\nabla T$, and hence we can identify

$$K = \frac{5}{2}\frac{n\tau_K k_B^2 T}{m}.$$

(e) What is the temperature profile, $T(y)$, of the gas in steady state?

- Since $\partial_t T$ is proportional to $-\partial_y h_y$, there will be no time variation if h_y is a constant. But $h_y = -K\partial_y T$, where K, which is proportional to the product nT, is a constant in the situation under investigation. Hence $\partial_y T$ must be constant, and $T(y)$ varies linearly between the two plates. Subject to the boundary conditions of $T(0) = T_1$, and $T(w) = T_2$, this gives

$$T(y) = T_1 + \frac{T_2 - T_1}{w}y.$$

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1. *Hard core gas:* A gas obeys the equation of state $P(V - Nb) = Nk_BT$, and has a heat capacity C_V independent of temperature. (N is kept fixed in the following.)

(a) Find the Maxwell relation involving $\partial S/\partial V|_{T,N}$.

• For $dN = 0$,

$$d(E - TS) = -SdT - PdV, \quad \Longrightarrow \quad \left.\frac{\partial S}{\partial V}\right|_{T,N} = \left.\frac{\partial P}{\partial T}\right|_{V,N}.$$

(b) By calculating $dE(T, V)$, show that E is a function of T (and N) only.

• Writing dS in terms of dT and dV,

$$dE = TdS - PdV = T\left(\left.\frac{\partial S}{\partial T}\right|_{V,N} dT + \left.\frac{\partial S}{\partial V}\right|_{T,N} dV\right) - PdV.$$

Using the Maxwell relation from part (a), we find

$$dE(T, V) = T\left.\frac{\partial S}{\partial T}\right|_{V,N} dT + \left(T\left.\frac{\partial P}{\partial T}\right|_{V,N} - P\right) dV.$$

But from the equation of state, we get

$$P = \frac{Nk_BT}{(V - Nb)}, \quad \Longrightarrow \quad \left.\frac{\partial P}{\partial T}\right|_{V,N} = \frac{P}{T}, \quad \Longrightarrow \quad dE(T, V) = T\left.\frac{\partial S}{\partial T}\right|_{V,N} dT,$$

i.e. $E(T, N, V) = E(T, N)$ does not depend on V.

(c) Show that $\gamma \equiv C_P/C_V = 1 + Nk_B/C_V$ (independent of T and V).

• The heat capacity is

$$C_P = \left.\frac{\partial Q}{\partial T}\right|_P = \left.\frac{\partial E + P\partial V}{\partial T}\right|_P = \left.\frac{\partial E}{\partial T}\right|_P + P\left.\frac{\partial V}{\partial T}\right|_P.$$

But, since $E = E(T)$ only,

$$\left.\frac{\partial E}{\partial T}\right|_P = \left.\frac{\partial E}{\partial T}\right|_V = C_V,$$

and from the equation of state we get

$$\left.\frac{\partial V}{\partial T}\right|_P = \frac{Nk_B}{P}, \quad \Longrightarrow \quad C_P = C_V + Nk_B, \quad \Longrightarrow \quad \gamma = 1 + \frac{Nk_B}{C_V},$$

which is independent of T, since C_V is independent of temperature. The independence of C_V from V also follows from part (a).

(d) By writing an expression for $E(P,V)$, or otherwise, show that an adiabatic change satisfies the equation $P(V - Nb)^\gamma =$ constant.

• Using the equation of state, we have

$$dE = C_V\, dT = C_V\, d\left(\frac{P(V - Nb)}{Nk_B}\right) = \frac{C_V}{Nk_B}\left(PdV + (V - Nb)dP\right).$$

The adiabatic condition, $dQ = dE + PdV = 0$, can now be written as

$$0 = dQ = \left(1 + \frac{C_V}{Nk_B}\right)Pd(V - Nb) + \frac{C_V}{Nk_B}(V - Nb)dP.$$

Dividing by $C_V P(V - Nb)/(Nk_B)$ yields

$$\frac{dP}{P} + \gamma\frac{d(V - Nb)}{(V - Nb)} = 0, \quad \Longrightarrow \quad \ln\left[P(V - Nb)^\gamma\right] = constant.$$

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2. *Energy of a gas:* The probability density to find a particle of momentum $\mathbf{p} \equiv (p_x, p_y, p_z)$ in a gas at temperature T is given by

$$p(\mathbf{p}) = \frac{1}{(2\pi mk_BT)^{3/2}}\exp\left(-\frac{p^2}{2mk_BT}\right), \qquad \text{where} \qquad p^2 = \mathbf{p}\cdot\mathbf{p} \quad .$$

(a) Using Wick's theorem, or otherwise, calculate the averages $\langle p^2\rangle$ and $\langle(\mathbf{p}\cdot\mathbf{p})(\mathbf{p}\cdot\mathbf{p})\rangle$.
• From the Gaussian form we obtain $\langle p_\alpha p_\beta\rangle = mk_BT\delta_{\alpha\beta}$, where α and β label any of the three components of the momentum. Therefore:

$$\langle p^2\rangle = \langle p_\alpha p_\alpha\rangle = mk_BT\delta_{\alpha\alpha} = 3mk_BT,$$

and using Wick's theorem

$$\langle(\mathbf{p}\cdot\mathbf{p})(\mathbf{p}\cdot\mathbf{p})\rangle = \langle p_\alpha p_\alpha p_\beta p_\beta\rangle = (mk_BT)^2\left[\delta_{\alpha\alpha}\delta_{\beta\beta} + 2\delta_{\alpha\beta}\delta_{\alpha\beta}\right] = 15\left(mk_BT\right)^2.$$

(b) Calculate the characteristic function for the energy $\varepsilon = p^2/2m$ of a gas particle.
• The characteristic function ε is the average $\langle e^{ik\varepsilon}\rangle$, which is easily calculated by Gaussian integration as

$$\langle e^{ik\varepsilon}\rangle = \langle e^{ikp^2/2m}\rangle = \int\frac{d^3\mathbf{p}}{(2\pi mk_BT)^{3/2}}\exp\left[\left(ik - \frac{1}{k_BT}\right)\frac{p^2}{2m}\right] = (1 - ikk_BT)^{-3/2}.$$

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(c) Using the characteristic function, or otherwise, calculate the mth cumulant of the particle energy $\langle \varepsilon^m \rangle_c$.

• The cumulants are obtained from the expansion

$$\ln \left\langle e^{ik\varepsilon} \right\rangle = \sum_{m=1}^{\infty} \frac{(ik)^m}{m!} \langle \varepsilon^m \rangle_c = -\frac{3}{2} \ln \left(1 - ikk_BT\right) = \frac{3}{2} \sum_{m=1}^{\infty} \frac{(k_BT)^m}{m} (ik)^m,$$

as

$$\langle \varepsilon^m \rangle_c = \frac{3}{2}(m-1)! \, (k_BT)^m \, .$$

(d) The total energy of a gas of N (independent) particles is given by $E = \sum_{i=1}^{N} \varepsilon_i$, where ε_i is the kinetic energy of the ith particle, as given above. Use the central limit theorem to compute the probability density for energy, $p(E)$, for $N \gg 1$.

• Since the energy E is the sum of N identically distributed independent variables, its cumulants are simply N times those for a signle variable, i.e.

$$\langle E^m \rangle_c = N \langle \varepsilon^m \rangle_c = \frac{3}{2} N(m-1)! \, (k_BT)^m \, .$$

According to the central limit theorem, in the large N limit the mean and variance are sufficient to describe the probability density, which thus assumes the Gaussian form

$$p(E) = \frac{1}{\sqrt{3\pi Nk_BT}} \exp \left[-\frac{(E - 3Nk_BT/2)^2}{3Nk_BT} \right].$$

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3. *'Relativistic' gas:* Consider a gas of particles with a 'relativistic' one particle Hamiltonian $\mathcal{H}_1 = c|\mathbf{p}|$, where $|\mathbf{p}| = \sqrt{p_x^2 + p_y^2 + p_z^2}$ is the magnitude of the momentum. (The external potential is assumed to be zero, expect at the edges of the box confining the gas particles.) Throughout this problem treat the two body interactions and collisions precisely as in the case of classical particles considered in lectures.

(a) Write down the Boltzmann equation for the one-particle density $f_1(\mathbf{p}, \mathbf{q}, t)$, using the same collision form as employed in lectures (without derivation).

• The Boltzmann equation has the general form

$$\mathcal{L}f_1 = C[f_1, f_1].$$

The collision term is assumed to be the same as in the classical case derived in lectures, and thus given by

$$C[f_1, f_1] = -\int d^3\mathbf{p_2} d^2\mathbf{b}|\mathbf{v_2} - \mathbf{v_1}| \left[f_1(\mathbf{p_1})f_1(\mathbf{p_2}) - f_1(\mathbf{p_1'})f_1(\mathbf{p_2'}) \right].$$

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(There are various subtleties in treatment of relativistic collisions, such as the meaning of $|\mathbf{v_2} - \mathbf{v_1}|$, which shall be ignored here.) The streaming terms have the form

$$\mathcal{L}f_1(\mathbf{p}, \mathbf{q}, t) = \partial_t f_1 + \{\mathcal{H}_1, f_1\} = \left[\partial_t + \frac{\partial \mathcal{H}_1}{\partial p_\alpha} \partial_\alpha\right] f_1 = \left[\partial_t + c\frac{p_\alpha}{|\mathbf{p}|}\partial_\alpha\right] f_1.$$

(b) The two body collisions conserve the number of particles, the momentum, and the particle energies as given by $\mathcal{H}_1$. Write down the most general form $f_1^0(\mathbf{p}, \mathbf{q}, t)$ that sets the collision integrand in the Boltzmann equation to zero. (You do not need to normalize this solution.)

• The integrand in $C[f_1, f_1]$ is zero if at each $\mathbf{q}$, $\ln f_1(\mathbf{p_1}) + \ln f_1(\mathbf{p_2}) = \ln f_1(\mathbf{p_1'}) + \ln f_1(\mathbf{p_2'})$. This can be achieved if $\ln f_1 = \sum_\mu a_\mu(\mathbf{q}, t)\chi_\mu(\mathbf{p})$, where $\chi_\mu(\mathbf{p})$ are quantities conserved in a two body collision, and a_μ are functions independent of $\mathbf{p}$. In our case, the conserved quantities are 1 (particle number), $\mathbf{p}$ (momentum), and $c|\mathbf{p}|$ (energy), leading to

$$f_1^0(\mathbf{p}, \mathbf{q}, t) = \exp\left[-a_0(\mathbf{q}, t) - \mathbf{a_1}(\mathbf{q}, t)\cdot\mathbf{p} - a_2(\mathbf{q}, t)c|\mathbf{p}|\right].$$

For any function $\chi(\mathbf{p})$ which is conserved in the collisions, there is a hydrodynamic equation of the form

$$\partial_t\left(n\langle\chi\rangle\right) + \partial_\alpha\left(n\left\langle c\frac{p_\alpha}{|\mathbf{p}|}\chi\right\rangle\right) - n\langle\partial_t\chi\rangle - n\left\langle c\frac{p_\alpha}{|\mathbf{p}|}\partial_\alpha\chi\right\rangle = 0,$$

where $n(\mathbf{q}, t) = \int d^3\mathbf{p} f_1(\mathbf{p}, \mathbf{q}, t)$ is the local density, and

$$\langle\mathcal{O}\rangle = \frac{1}{n}\int d^3\mathbf{p} f_1(\mathbf{p}, \mathbf{q}, t)\mathcal{O}.$$

(c) Obtain the equation governing the density $n(\mathbf{q}, t)$, in terms of the average local velocity $u_\alpha = \langle cp_\alpha/|\mathbf{p}|\rangle$.

• Substituting $\chi = 1$ in the conservation equation gives

$$\partial_t n + \partial_\alpha\left(nu_\alpha\right) = 0, \qquad \text{with} \qquad u_\alpha = \langle cp_\alpha/|\mathbf{p}|\rangle.$$

(d) Find the hydrodynamic equation for the local momentum density $\pi_\alpha(\mathbf{q}, t) \equiv \langle p_\alpha\rangle$, in terms of the pressure tensor $P_{\alpha\beta} = nc\langle(p_\alpha - \pi_\alpha)(p_\beta - \pi_\beta)/|\mathbf{p}|\rangle$.

• Since momentum is conserved in the collisions, we can obtain a hydrodynamic equation by putting $\chi_\alpha = p_\alpha - \pi_\alpha$ in the general conservation form. Since $\langle\chi_\alpha\rangle = 0$, this leads to

$$\partial_\beta\left(n\left\langle c\frac{\chi_\beta + \pi_\beta}{|\mathbf{p}|}\chi_\alpha\right\rangle\right) + n\partial_t\pi_\alpha + nu_\beta\partial_\beta\pi_\alpha = 0.$$

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Further simplification and rearrangements leads to□

$$D_t \pi_\alpha \equiv \partial_t \pi_\alpha + u_\beta \partial_\beta \pi_\alpha = -\frac{1}{n} \partial_\beta P_{\beta\alpha} - \frac{1}{n} \partial_\beta \left(n \pi_\beta c \left\langle \frac{\chi_\alpha}{|\mathbf{p}|} \right\rangle \right).$$

(Unfortunately, as currently formulated, the problem does not lead to a clean answer, in that there is a second term in the above result that does not depend on $P_{\alpha\beta}$.)

(e) Find the (normalized) one particle density $f_1(\mathbf{p}, \mathbf{q}, t)$ for a gas of N such particles in a box of volume V, in equilibrium at a temperature T.

• At equilibrium, the temperature T and the density $n = N/V$ are uniform across the system, and there is no local velocity. The general form obtained in part (b) now gives

$$f_1^0(\mathbf{p}, \mathbf{q}, t) = \frac{N}{V} \exp\left(-\frac{c|\mathbf{p}|}{k_B T} \right) \frac{1}{8\pi} \left(\frac{c}{k_B T} \right)^3.$$

The normalization factor is obtained by requiring $N = V \int d^3\mathbf{p} f_1$, noting that $d^3\mathbf{p} = 4\pi p^2 dp$, and using $\int_0^\infty dp\, p^n e^{-p/a} = n! a^{n+1}$.

(f) Evaluate the pressure tensor $P_{\alpha\beta}$ for the above gas in equilibrium at temperature T.

• For the gas at equilibrium $\pi_\alpha = u_\alpha = 0$, and the pressure tensor is given by

$$P_{\alpha\beta} = nc \left\langle \frac{p_\alpha p_\beta}{|\mathbf{p}|} \right\rangle = nc\delta_{\alpha\beta} \left\langle \frac{p_x p_x}{|\mathbf{p}|} \right\rangle = \delta_{\alpha\beta} \frac{nc}{3} \left\langle \frac{\mathbf{p} \cdot \mathbf{p}}{|\mathbf{p}|} \right\rangle.$$

In rewriting the above equation we have taken advantage of the rotational symmetry of the system. The expectation value is simply

$$\langle |\mathbf{p}| \rangle = \frac{\int_0^\infty dp\, p^2 p e^{-cp/k_B T}}{\int_0^\infty dp\, p^2 e^{-cp/k_B T}} = 3\frac{k_B T}{c},$$

leading to

$$P_{\alpha\beta} = \delta_{\alpha\beta} n k_B T,$$

which is the usual formula for an ideal gas.

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1. *Wire:* Experiments on stretching an elastic wire indicate that, at a temperature T, a displacement x requires a force

$$J = ax - bT + cTx,$$

where a, b, and c are constants. Furthermore, its heat capacity at constant displacement is proportional to temperature, i.e. $C_x = A(x)T$.

(a) Use an appropriate Maxwell relation to calculate $\partial S/\partial x|_T$.

• From $dF = -SdT + Jdx$, we obtain

$$\left.\frac{\partial S}{\partial x}\right|_T = -\left.\frac{\partial J}{\partial T}\right|_x = b - cx.$$

(b) Show that A has to be independent of x, i.e. $dA/dx = 0$.

• We have $C_x = T\left.\frac{\partial S}{\partial T}\right|_x = A(x)T$, where $S = S(T, x)$. Thus

$$\frac{\partial A}{\partial x} = \frac{\partial}{\partial x}\frac{\partial S}{\partial T} = \frac{\partial}{\partial T}\frac{\partial S}{\partial x} = 0$$

from part (a), implying that A is independent of x.

(c) Give the expression for $S(T, x)$, and comment on whether it is compatible with the third law of thermodynamics.

• By integrating the derivatives of S given above, $S(x, T)$ can be calculated as

$$S(x, T) = S(0, 0) + \int_{T'=0}^{T'=T} \frac{\partial S(T', x = 0)}{\partial T'}dT' + \int_{x'=0}^{x'=x} \frac{\partial S(T, x')}{\partial x'}dx$$

$$= S(0, 0) + \int_0^T AdT' + \int_0^x (b - cx')dx'$$

$$= S(0, 0) + AT + bx - \frac{c}{2}x^2.$$

However, $S(T = 0, x) = S(0, 0) + bx - cx^2/2$, now explicitly depends on x, in violation of the third law of thermodynamics.

(d) Calculate the heat capacity at constant tension, i.e. $C_J = T\,\partial S/\partial T|_J$, as a function of T and J.

• Writing the entropy as $S(T, x) = S(T, x(T, J))$, leads to

$$\left.\frac{\partial S}{\partial T}\right|_J = \left.\frac{\partial S}{\partial T}\right|_x + \left.\frac{\partial S}{\partial x}\right|_T \left.\frac{\partial x}{\partial T}\right|_J.$$

From parts (a) and (b), $\frac{\partial S}{\partial x}\big|_T = b - cx$ and $\frac{\partial S}{\partial T}\big| x = A$. Furthermore, $\frac{\partial x}{\partial T}\big|_J$ is given by $a\frac{\partial x}{\partial T} - b + cx + cT\frac{\partial x}{\partial T} = 0$, i.e.

$$\frac{\partial x}{\partial T} = \frac{b - cx}{a + cT}.$$

Thus

$$C_J = T\left[A + \frac{(b - cx)^2}{(a + cT)}\right].$$

Since $x = \frac{J + bT}{a + cT}$, we can rewrite the heat capacity as a function of T and J, as

$$C_J = T\left[A + \frac{(b - c\frac{J+bT}{a+cT})^2}{(a + cT)}\right]$$

$$= T\left[A + \frac{(ab - cJ)^2}{(a + cT)^3}\right].$$

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2. *Random matrices:* As a model for energy levels of complex nuclei, Wigner considered $N \times N$ symmetric matrices whose elements are random. Let us assume that each element M_{ij} (for $i \geq j$) is an independent random variable taken from the probability density function

$$p(M_{ij}) = \frac{1}{2a} \quad \text{for} \quad -a < M_{ij} < a, \quad \text{and} \quad p(M_{ij}) = 0 \quad \text{otherwise.}$$

(a) Calculate the characteristic function for each element M_{ij}.
● Since each element is uniformly distributed in the interval $(-a, a)$, the characteristic function is

$$\tilde{p}_{ij}(k) = \int_{-a}^{a} dx \ e^{-ikx} \frac{1}{2a} = \frac{e^{ika} - e^{-ika}}{2aik} = \frac{\sin ak}{ak}.$$

(b) Calculate the characteristic function for the trace of the matrix, $T \equiv \operatorname{tr} M = \sum_i M_{ii}$.
● The trace of the matrix is the sum of the N diagonal elements which are independent random variables. The characteristic function for the sum of independent variables is simply the product of the corresponding characteristic functions, and thus

$$\tilde{p}_T(k) = \prod_{i=1}^{N} \tilde{p}_{ii}(k) = \left(\frac{\sin ak}{ak}\right)^N.$$

(c) What does the central limit theorem imply about the probability density function of the trace at large N?

- Since the trace is the sum of $N \gg 1$ independent random variables, its cumulants are simply N times those of a single element. The leading cumulants are

$$\langle T \rangle_c = N \langle M_{ij} \rangle = 0,$$

and

$$\langle T^2 \rangle_c = N \langle M_{ij}^2 \rangle = N \int_{-a}^{a} dx \, \frac{x^2}{2a} = N \frac{a^2}{3}.$$

For the qunatity $t = T/\sqrt{N}$, higher order cumulants vanish in the limit of $N \to \infty$, and thus

$$\lim_{N \to \infty} p \left(t = \frac{\text{tr} \, M}{\sqrt{N}} \right) = \exp \left(-\frac{3t^2}{2a^2} \right) \sqrt{\frac{3}{2\pi a^2}}.$$

(d) For large N, each eigenvalue λ_α ($\alpha = 1, 2, \cdots, N$) of the matrix M is distributed according to a probability density function

$$p(\lambda) = \frac{2}{\pi \lambda_0} \sqrt{1 - \frac{\lambda^2}{\lambda_0^2}} \quad \text{for} \quad -\lambda_0 < \lambda < \lambda_0, \quad \text{and} \quad p(\lambda) = 0 \quad \text{otherwise,}$$

(known as the Wigner semi-circle rule). Find the variance of λ.
(**Hint:** Changing variables to $\lambda = \lambda_0 \sin \theta$ simplifies the integrals.)
- The mean value of λ is zero by symmetry, and hence its variance is given by

$$\langle \lambda^2 \rangle_c = \int_{-\lambda_0}^{\lambda_0} d\lambda \, \lambda^2 \frac{2}{\pi \lambda_0} \sqrt{1 - \frac{\lambda^2}{\lambda_0^2}}.$$

In the integral, change variables to $\lambda = \lambda_0 \sin \theta$ and $d\lambda = \lambda_0 \cos \theta d\theta$, to get

$$\langle \lambda^2 \rangle_c = \frac{2\lambda_0^2}{\pi} \int_{-\pi/2}^{\pi/2} d\theta \, \cos^2 \theta \sin^2 \theta = \frac{\lambda_0^2}{2\pi} \int_{-\pi/2}^{\pi/2} d\theta \, \sin^2 2\theta = \frac{\lambda_0^2}{4\pi} \int_{-\pi/2}^{\pi/2} d\theta \, (1 - \cos 4\theta) = \frac{\lambda_0^2}{4}.$$

(e) If in the previous result, we have $\lambda_0^2 = 4Na^2/3$, can the eigenvalues be independent of each other?
- The trace of a matrix is related to its eigenvalues by

$$T = \sum_{\alpha=1}^{N} \lambda_\alpha, \quad \Longrightarrow \quad \langle T^2 \rangle_c = \sum_{\alpha=1}^{N} \langle \lambda_\alpha^2 \rangle + \sum_{\alpha \neq \beta} \langle \lambda_\alpha \lambda_\beta \rangle.$$

The cross-correlations of eigenvalues thus satisfy

$$\sum_{\alpha \neq \beta} \langle \lambda_\alpha \lambda_\beta \rangle = \langle T^2 \rangle_c - \sum_{\alpha=1}^{N} \langle \lambda_\alpha^2 \rangle = N \frac{a^2}{3} - N \times \frac{4Na^2}{3} \neq 0.$$

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Clearly, this is inconsistent with independent eigenvalues. In fact, the well known result that eigenvalues do not cross implies a repulsion between eigenvalues which leads to a much wider distribution than would result from independent eigenvalues.

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3. *Viscosity:* Consider a classical gas between two plates separated by a distance w. One plate at $y = 0$ is stationary, while the other at $y = w$ moves with a constant velocity $v_x = u$. A zeroth order approximation to the one particle density is,

$$f_1^0(\vec{p}, \vec{q}) = \frac{n}{(2\pi m k_B T)^{3/2}} \exp\left[-\frac{1}{2m k_B T}\left((p_x - m\alpha y)^2 + p_y^2 + p_z^2\right)\right],$$

obtained from the *uniform* Maxwell–Boltzmann distribution by substituting the average value of the gas velocity at each point. ($\alpha = u/w$ is the velocity gradient, while n and T are constants.)

(a) The above approximation does not satisfy the Boltzmann equation as the collision term (right hand side of the equation) vanishes, while (the left hand side) $df_1^0/dt \neq 0$. Find a better approximation, $f_1^1(\vec{p})$, by considering the linearized Boltzmann equation in the single collision time approximation, i.e.

$$\mathcal{L}\left[f_1^1\right] \approx \left[\frac{\partial}{\partial t} + \frac{\vec{p}}{m} \cdot \frac{\partial}{\partial \vec{q}}\right] f_1^0 \approx -\frac{f_1^1 - f_1^0}{\tau_\times},$$

where $\tau_\times$ is a characteristic mean time between collisions.

• We have

$$\left(\frac{\partial}{\partial t} + \frac{\vec{p}}{m} \cdot \frac{\partial}{\partial \vec{q}}\right) f_1^0 = \frac{\alpha}{m k_B T} p_y (p_x - m\alpha y) f_1^0,$$

whence

$$f_1^1 = f_1^0 \left\{1 - \tau_\times \frac{\alpha}{m k_B T} p_y (p_x - m\alpha y)\right\}.$$

(b) Calculate the off-diagonal component $P_{xy}(y)$ of the pressure tensor.

• The pressure tensor is $P_{\alpha\beta}(y) = nm \langle c_\alpha c_\beta \rangle = n \langle p_\alpha p_\beta \rangle /m$. From the first order density, the off-diagonal element is calculated as

$$P_{xy}(y) = \int d^3p \frac{p_x p_y}{m} f_1^1(y)$$

$$= \int d^3p \frac{p_x p_y}{m} \left(-\frac{\tau_\times \alpha}{m k_B T}\right) p_y (p_x - m\alpha y) f_1^0$$

$$= \frac{-\tau_\times \alpha n}{m^2 k_B T} \left\{\int dp_x (p_x - m\alpha y)^2 \frac{\exp\left(-\frac{(p_x - m\alpha y)^2}{2m k_B T}\right)}{\sqrt{2\pi m k_B T}}\right\} \left\{\int dp_y p_y^2 \frac{\exp\left(-\frac{p_y^2}{2m k_B T}\right)}{\sqrt{2\pi m k_B T}}\right\}$$

$$= -\frac{\tau_\times \alpha n}{m^2 k_B T} (m k_B T)^2 = -\alpha n \tau_\times k_B T.$$

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(c) The gas exerts a transverse force per unit area $F_x = -P_{xy}(y = w)$ on the moving plate. Calculate this force, and hence obtain the coefficient of viscosity, defined by $\eta = F_x/\alpha$.

• The pressure tensor calculated in part (b) is in fact independent of the position y, and the force exerted on the top plate (or the bottom plate) is thus

$$F_x = -P_{xy} = \alpha n \tau_\times k_B T.$$

The coefficient of viscosity is then simply

$$\eta = \frac{F_x}{\alpha} = n \tau_\times k_B T.$$

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