Solitons: Conservation Laws and the Lax Formulation.

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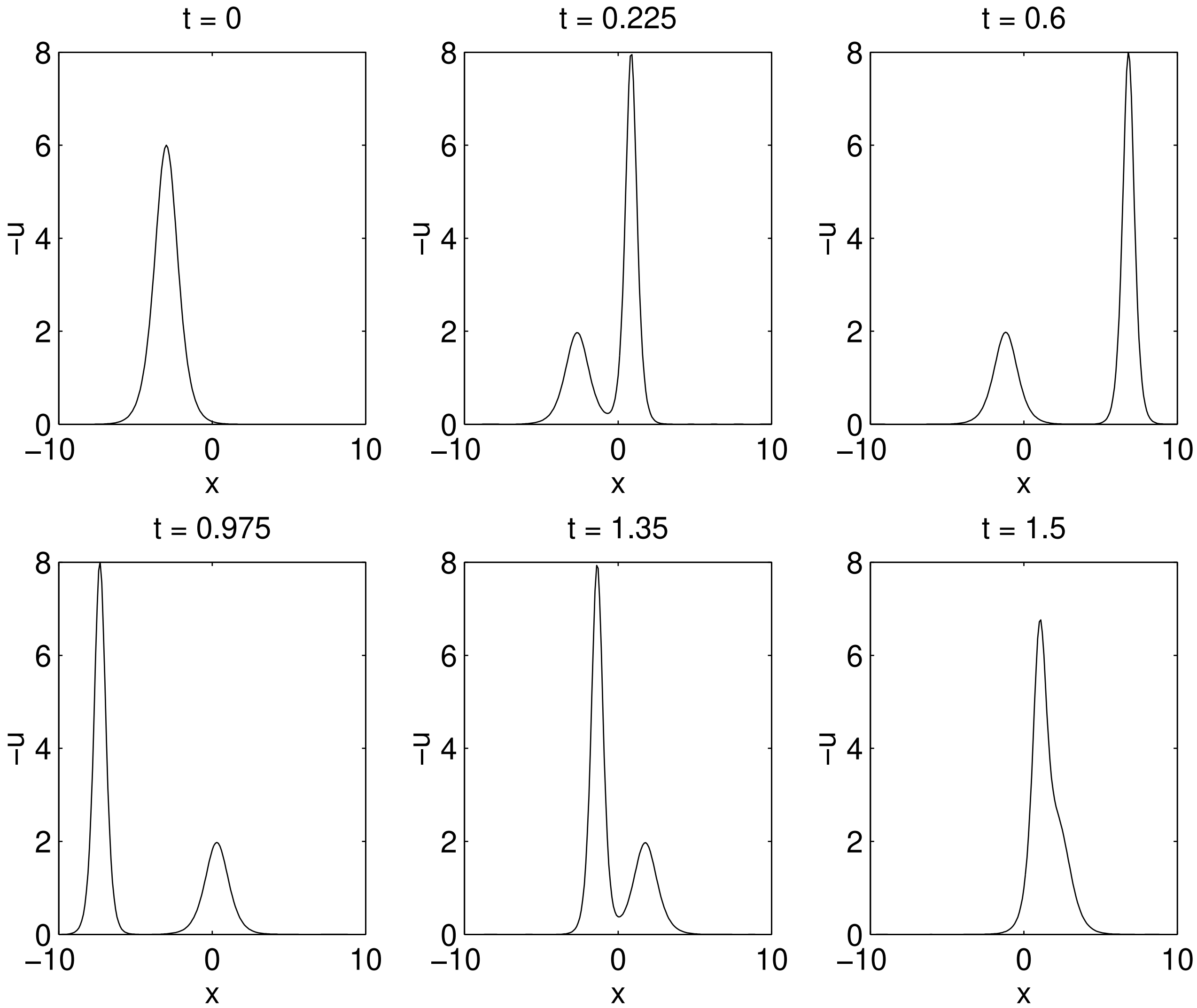
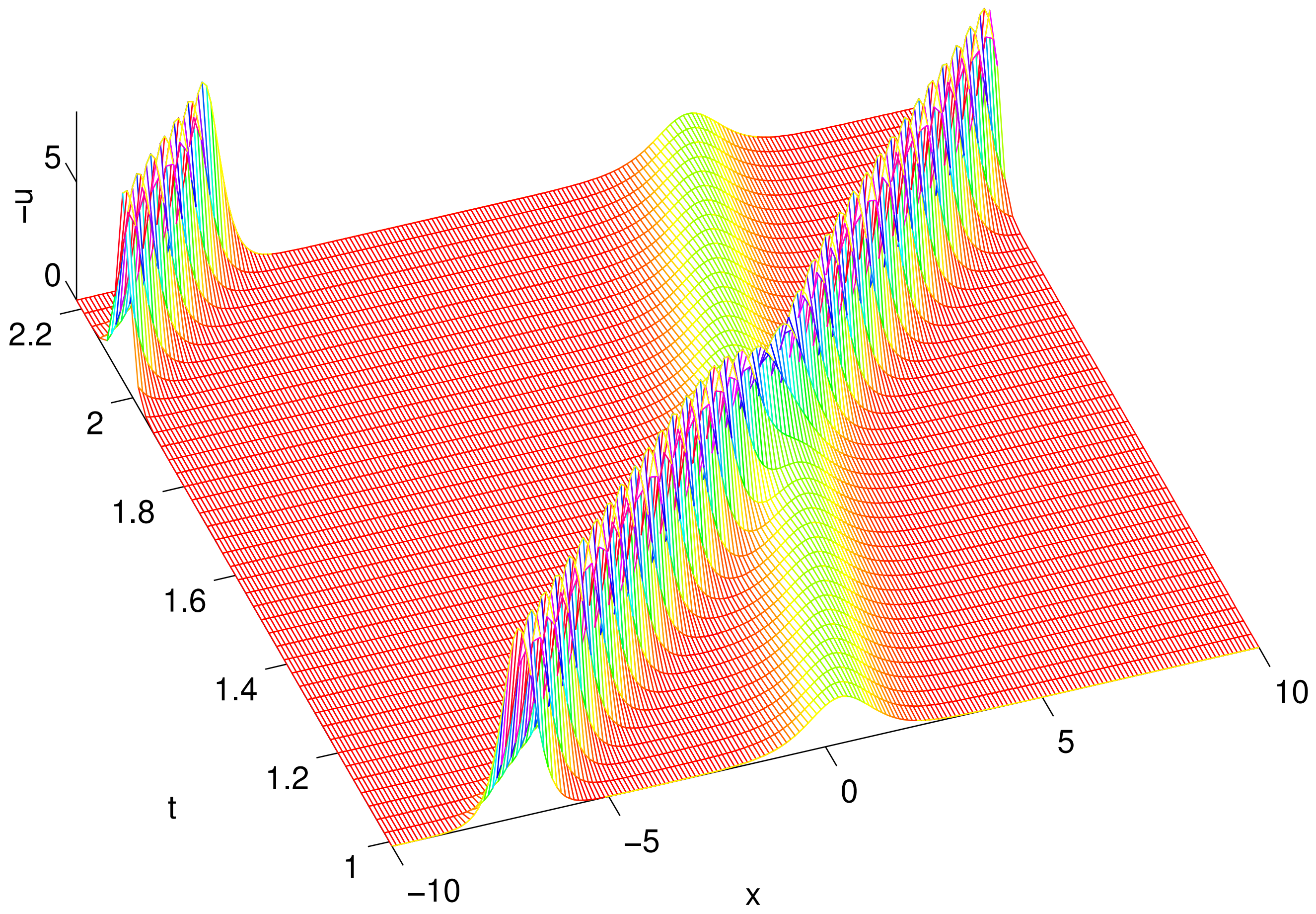
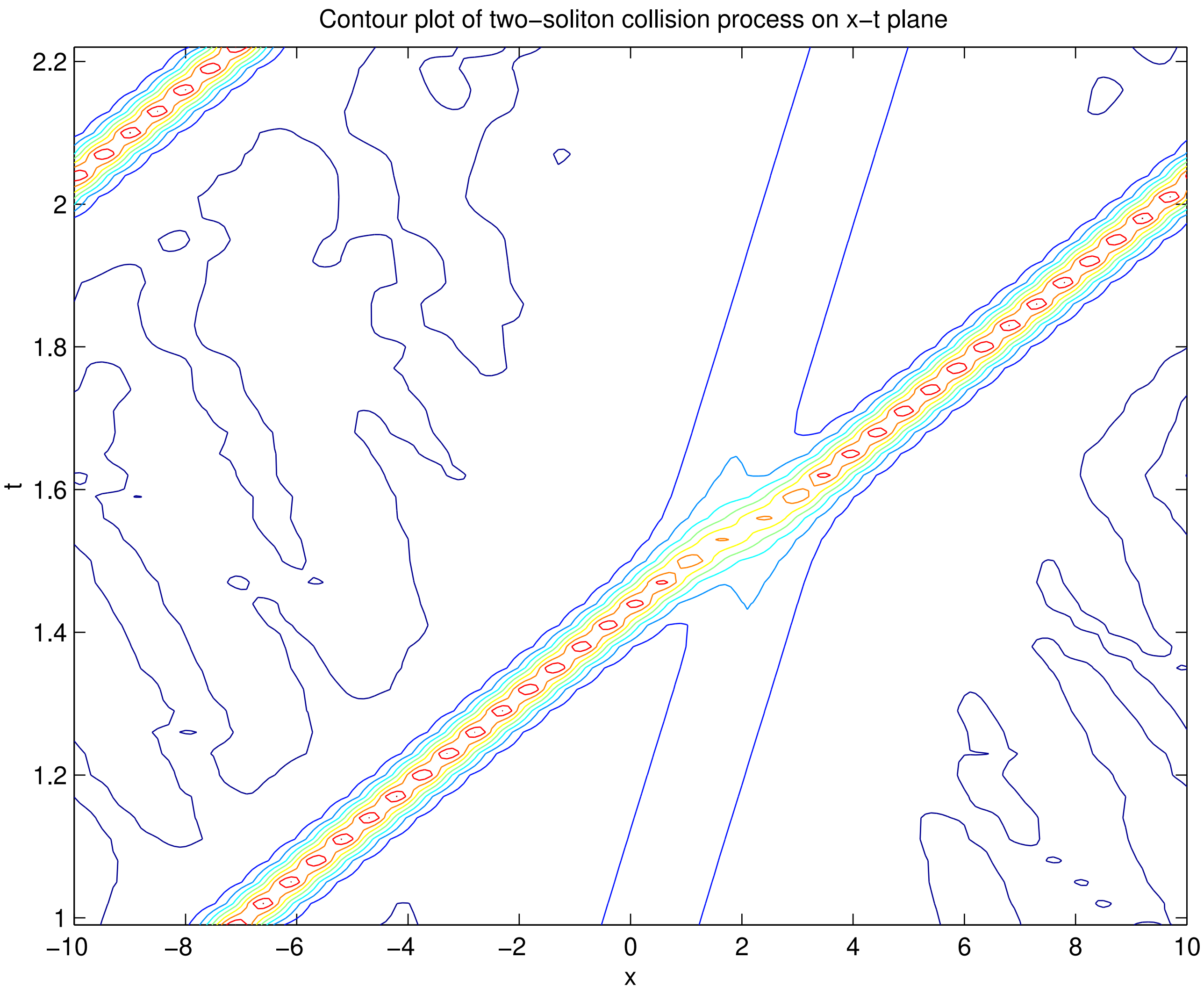


t = 0
t = 0.225
t = 0.6
t = 0.975
t = 1.35
t = 1.5
−u
x
0
2
4
6
8
−10
0
10

Two−soliton (KdV) collision process shown with x−t plot




Contour plot of two−soliton collision process on x−t plane
t
x
2.2
2
1.8
1.6
1.4
1.2
1
−10
−8
−6
−4
−2
0
2
4
6
8
10

Soliton equations:

- Korteweg-de Vries equation (shallow water waves, anharmonic crystal):

$$u_t - 6uu_x + u_{xxx} = 0$$

- sine-Gordon equation (dislocation mobility, Josephson-junction):

$$u_{tt} - u_{xx} + \sin(u) = 0$$

- nonlinear Schrodinger equation (amplitude equations, nonlinear optics):

$$iA_t + A_{xx} \pm A|A|^2 = 0$$

- higher-order equations:

$$u_t + 30u^2u_x - 20u_xu_{xx} - 10uu_{xxx} + u_{xxxxx} = 0 \qquad \text{(Lax hierarchy)}$$
$$u_t + 45u^2u_x - 15u_xu_{xx} - 15uu_{xxx} + u_{xxxxx} = 0 \qquad \text{(bilinear hierarchy)}$$

Intuitive definition of solitons: Any solution of a nonlinear equation which

1. represents a wave of permanent form.
2. is localized, so that it approaches a constant at $\pm\infty$.
3. interacts strongly with other solitons and retains its identity.

Inverse Scattering Transform: Let $u(x)$ be the *potential function* of the Schrodinger scattering problem:

$$\psi_{xx} + (\lambda - u)\psi = 0 \qquad\qquad -\infty < x < +\infty$$

then u can be *reconstructed* from the scattering data (asymptotic behaviour of the eigenfunction ψ) in the form of

$$\psi(x;k) \sim \begin{cases} e^{-ikx} + b(k)e^{ikx} & \text{as} \quad x \to +\infty \\ a(k)e^{-ikx} & \text{as} \quad x \to -\infty \end{cases}$$

with $k^2 = \lambda > 0$ for the continuous spectrum, and

$$\psi_n(x) \sim c_n \exp(-\kappa_n x) \qquad \text{as} \quad x \to +\infty$$

with $\kappa_n^2 = -\lambda > 0$ for each discrete eigenvalue $(n = 1, 2, .., N)$ and c_n picked to normalize ψ_n. It can be proved that

$$u(x) = -2\frac{d}{dx}K(x,x)$$

where $K(x,z)$ is the solution to the Marchenko equation

$$K(x,z) + F(x+z) + \int_x^{+\infty} K(x,y)F(y+z)dy = 0$$

with F defined by

$$F(x) = \sum_{n=1}^{N} c_n^2 \exp(-\kappa_n x) + \frac{1}{2\pi}\int_{-\infty}^{+\infty} b(k)e^{ikx}dk$$

Soliton Magic: for the KdV equation

$$u_t - 6uu_x + u_{xxx} = 0$$

It can be shown, through quite laborious work, that the following is true:

If u(x,t) evolves according to the above nonlinear equation, then

1. *The eigen-spectra of the Schrodinger scattering problem remain unchanged.*
2. $c_n(t) = c_n(0) exp(4\kappa_n^3 t)$
3. $a(k;t) = a(k;0), \qquad b(k;t) = b(k;0) exp(8ik^3 t)$

It means we can solve the initial-value problem $u(x,0) = f(x)$ of a special class of nonlinear PDE's by first solving a quantum scattering problem under potential function $f(x)$, get $\kappa_n, c_n(0), b(k;0)$, multiply by the simple time factors, then inverse transform back to $u(x,t)$ through IST. In a sense it's very similiar to the Fourier transform method in solving linear PDE's, however we have reduced the solving of a nonlinear PDE into solving two *linear* equations. It also gives a more fundamental definition to soliton:

A soliton is that component of the solution of a nonlinear evolution equation which depends only upon one constant discrete eigenvalue of the underlying scattering problem as $t \to \pm\infty$.

Soliton Magic : $u_t - 6uu_x + u_{xxx} = 0$

- Conservation Laws: If we have equation(s) of the form

$$\frac{\partial T}{\partial t} + \frac{\partial X}{\partial x} = 0$$

then we have a conservation law and T is called a *conserved density.* The following quantities

$$T_1 = u$$
$$T_2 = u^2$$
$$T_3 = u^3 + u_x^2/2$$

are conserved densities, corresponding to the conservation of mass, momentum and energy in the original shallow-water waves problem.

It came as a big surprise though that there are many other conserved densities, as discovered by Miura, Gardner and Kruskal(1968). They found *eight* more:

$$T_4 = 5u^4 + 10uu_x^2 + u_{xx}^2$$
$$T_5 = 21u^5 + 105u^2u_x^2 + 21uu_{xx}^2 + u_{xxx}^2$$
$$\dots$$