In the next two lectures (Oct.16, Oct.18), we will discuss some basics of fracture mechanics using continuum theories. The method of continuum mechanics is to view a solid as a continuous distribution of material particles. Each material particle consists of many atoms. The material particle represents the collective behavior of many atoms. Following topics will be discussed.
- Trouble with linear elastic theory of strength.
- The Griffith approach.
- Fracture energy. Energy release rate.
- Applications of fracture mechanics.

The problem to be solved in fracture mechanics. A body is subject to a load. What is the magnitude of the load that will cause the body to fracture? Let us begin with a body of a silica glass, which deforms elastically by small strains. A procedure has been taught in strength of materials probably goes as follows. We first determine the maximum stress in the body. Then we determine the strength of the material. The body is supposed to fracture when the maximum stress in the body reaches the strength of the material.

I will first review this procedure, so that we agree exactly what this procedure is. I will then explain why this procedure is difficult to apply in practice.

Theory of linear elasticity. I assume you have learned the elements of linear elasticity. Imagine a body subject to an applied stress, σ_{appl}. At each material particle, the state of stress is a tensor, with 6 components. In the body, the state of stress varies from one material particle to another. Thus, the state of stress in a body is described by a field. The field is determined by solving the boundary-value problem. A list of notes of solving boundary-value problems in elasticity are following:
- Governing equations. Geometric relations. Force balance equations. Constitutive law.
- Boundary conditions. Traction or displacement boundary conditions.
- Approximate solutions. Beams and plates.
- Exact analytical solutions. Few problems can be solved exactly. *Theory of Elasticity*, S. P. Timoshenko and J. N. Goodier, McGraw-Hill, New York.
- Numerical solutions. Finite element methods. Commercial software such as ABAQUS.

Maximum stress in a body. Solving boundary-value problem is a big task by itself, but is not the subject of this class. Let's say we already have the solution. That is, we know all six components of stress at every material particle in the body,

$$\sigma_{11}\left(x_1, x_2, x_3\right), \sigma_{12}\left(x_1, x_2, x_3\right)\ldots$$

What do we do with this massive amount of data? We are interested in

predicting the conditions of fracture of the body. For example, if the applied stress is small enough, the body will not fracture. How small is small enough?

Here is the procedure taught in strength of materials. From the field of stress we determine the maximum stress in the body, $\sigma_{\max}$. The maximum component of stress at each material particle is determined by principle stresses, solved by an eigenvalue problem. Then we look for the largest value of the principle stress by comparing all material particles in the body. Today, all this procedure is embodied in commercial finite element software. So you know the maximum component of stress of the body, $\sigma_{\max}$.

Stress concentration factor. The equations in elasticity are linear, so that the maximum stress in the body is proportional to the applied stress.

$$\sigma_{\max} = C\sigma_{\mathrm{appl}},$$

where C is a dimensionless number. The basic phenomenon that the stress is higher at some material particle in a body than others is known as stress concentration. The number C is known as the stress concentration factor.

When a circular hole is embedded in a much larger plate, this boundary-value problem is solved analytically by Timoshenko and Goodier. The maximum stress occurs at the surface of the hole. The stress concentration factor is

$$\frac{\sigma_{\max}}{\sigma_{\mathrm{appl}}} = 3.$$

Consider an elliptic hole in an infinite plate subject to a remote stress σ_{appl}. This boundary-value problem can still be solved analytically (*Stresses in a plate due to the presence of cracks and sharp corners,* C. E. Inglis, 1913). The maximum stress $\sigma_{\max}$ is given by

$$\frac{\sigma_{\max}}{\sigma_{\mathrm{appl}}} = 1 + \frac{2a}{b},$$

where a and b are semi-axes of the ellipse. The stress concentration factor depends on the shape of the hole, characterized by the ratio a/b. When $a=b$, the hole is circular, and the stress concentration factor is 3. When the ellipse is very elongated, $a>>b$, the stress concentration factor is very large.

Denote the radius of curvature at the tip of the ellipse as ρ, after some math you can find $\rho = b^2/a$. You can express the above formula in terms of a and ρ, namely,

$$\frac{\sigma_{\max}}{\sigma_{\mathrm{appl}}} \approx 2\sqrt{\frac{a}{\rho}}.$$

This formula may be used to estimate the stress concentration factor for a flaw of

some other shapes, where a is interpreted as the "overall size" of the flaw, and ρ is the radius at the root of the flaw.

Strength of a material. Now we have calculated the maximum stress σ_{max} in the body. Will the body sustain this stress? The theory of elasticity will not answer this question. You need to find the strength of the material somewhere else. For example, you may estimate the theoretical strength of a material by

$$S_{th} = \frac{E}{10}.$$

For a glass, Young's modulus is 70 GPa, so that the estimated theoretical strength is 7 GPa. Prof. Ju Li has told you this value is about two orders of magnitude higher than the strength measured in a bulk sample.

You can determine the strength of the material experimentally. You pull a sample until it breaks, then record the stress that breaks the sample. We call this stress as the experimental strength of the material. The experimental values of vlass are on the order of S ~100 MPa.

Design for strength based on linear elasticity. What is the maximum load that can be sustained by a body? We now summarize the procedure as follows.

• Calculate the stress field by solving boundary-value problems. Locate the maximum stress σ_{max} in the body.

• Assume the material has a definite strength. That is, the same material has the same strength, independent of the shape of the body. Measure the strength S using a simple sample of the material, such as a tensile bar.

• Make sure the maximum stress in the body is below the strength of the material.

Calculate stress field in the body. Locate σ_{max}

Measure the strength, S, of the material using a simple sample

Make sure $\sigma_{max} < S$
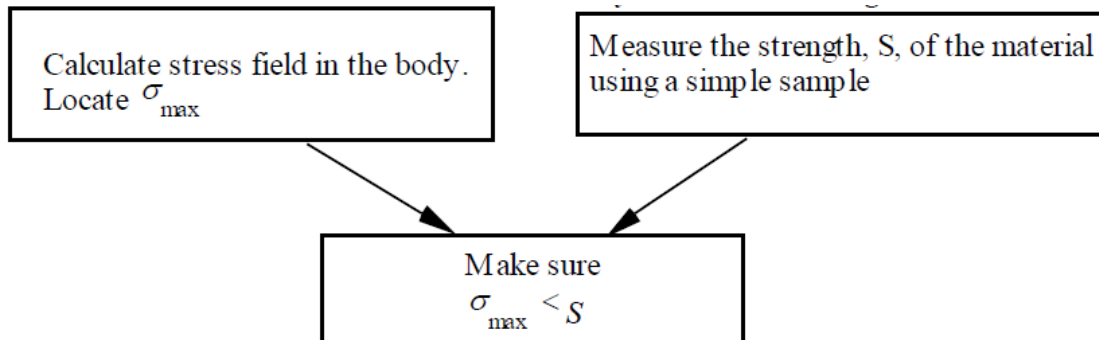
Why is this procedure hard to use in practice?
- The maximum stress in a body is sensitive to the shape of the flaw.
- The shape of the flaw is a body is seldom known in practice.
- The procedure assumes that the body is linearly elastic everywhere, which is never true.
- The procedure assumes that the strength of a material is independent of the sample used in experiments. In reality, strengths measured from different samples are different, because each sample has different flaws.

That is, the procedure is hit in both ways: the maximum stress is impossible to calculate, and the strength is impossible to measure.

Qualitative content of the Griffith paper. After the atomistic nature of matter was confirmed by many experimental observations, it became useful to relate macroscopic phenomena to atomic processes. In 1921, the British engineer A. A. Griffith published a paper on fracture of a glass. The main puzzle had been that the glass breaks under a stress several orders of magnitude below the strength of atomic cohesion. Griffith took up the notion that a piece of a glass is never perfect: small cracks pre-exist in a body of the glass. The tip of such a crack concentrates stresses. The intense stress at the tip breaks atomic bonds one by one, like opening a zipper. The crack advances, leading to the fracture of the body.

Consider a pre-existing crack in a body subject to an external load. Regard the body and the external force together as a thermodynamic system, characterized by two thermodynamic variables: the area of the crack and the displacement of the loading grips. To focus on the essential ideas, suppose that, after a certain amount of displacement, the loading grips are held fixed, but the crack is allowed to advance. Because the loading grips are fixed, the external load does no work. The energy of the system is the sum of the elastic energy in the body, and the surface energy in the faces of the crack. The energy of the system is the sum of the elastic energy in the body, and the surface energy in the faces of the crack. The energy of the system is a function of a single thermodynamic variable: the area of the crack. When the crack advances, the stress in the sample is partially relieved, so that the elastic energy is reduced. At the same time, the advancing crack creates more surface area, so that the surface energy increases. Thermodynamics dictates that the process should go in the direction that reduces the total free energy. If the decrease in elastic energy prevails, the crack grows. If the increase in surface energy prevails, the crack heals.

The nonlinear zone, localized around the tip of the crack, remains invariant as the crack advances. Consequently, the presence of the nonlinearity does not affect the accounting of energy.

To sum up, the qualitative content of the Griffith paper is following
- In a body of glass cracks pre-exist.
- The tip of such a crack concentrates stresses.
- The intense stress breaks atomic bonds one by one, like opening a zipper.
- As the crack advances, fresh surfaces are created. The surface energy increases, but the elastic energy decreases.
- The crack advances if the advance reduces the sum of the surface energy and elastic energy.

The theory. A large sheet of a glass is under stress σ. The sheet has unit thickness. For the time being, assume that the loading grips are rigidly held, so that the displacement is fixed, and the loading device does no additional work after a fixed displacement is applied. The state of reference is a stressed sheet with no crack. The state that interests us is the sheet with a crack of length $2a$. We now calculate the difference in energy between the two states.

The surface energy increases by $4a\gamma$.

The elastic energy reduces. To determine the amount of the reduction, one has to solve the boundary-value problem. This tough elasticity problem is for professional elastitians. Look how complicated the stress field must be near the crack. Griffith used the elasticity solution of Inglis, because a crack is just a special case of an ellipse when $b/a \to 0$. This part of the Griffith paper is difficult to read, and is not very interesting. In the end he made small errors. An alternative approach is to invoke linearity and dimensional considerations. For a linearly elastic problem, the stress field is linearly proportional to the applied stress. The elastic energy per unit volume is proportional to σ^2/E. The elastic energy in an infinite sheet is infinite. However, we are interested in the difference in elastic energy between the cracked sheet and the uncracked sheet. Note that the crack length a is the only length scale in the boundary-value problem. Consequently, the difference in elastic energy between the two sheets takes the form $\beta \dfrac{\sigma^2}{E} a^2$, where β is a numerical value. Thus, from very basic considerations, we get nearly everything except for a pure number. This number must be determined by solving the elasticity boundary value problem. The solution turns out to be $\beta = \pi$. You can find the solution of the full problem in Timoshenko and Goodier. Relative to the uncracked sheet, in the cracked sheet the combined surface energy and elastic energy is

$$U(a) = +4a\gamma - \pi \frac{\sigma^2}{E} a^2 .$$

The crack length, $2a$, is the thermodynamic variable. The surface energy density γ and the applied stress σ are taken to be constant for the time being. As expected, when the crack length increases, the surface energy increases, but the elastic energy decreases.

Critical crack size. Plot the free energy as a function of the crack length. The free energy first goes up, reaches a peak, and then goes down. Because there is no minimum free energy, the crack cannot reach equilibrium. The free energy reaches the peak at the crack length

$$a^* = \frac{2\gamma E}{\pi \sigma^2} .$$
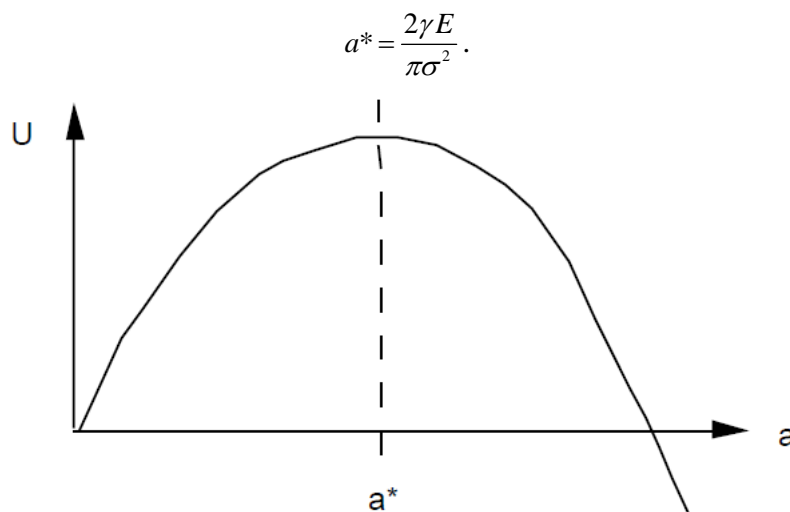
Let us examine the physical significance of this particular crack length. Let a be the length of the crack pre-existing in the sheet. Distinguish two situations.

Crack healing. If $a < a^*$, the surface energy prevails over the elastic energy. To reduce the free energy, the crack length must decrease. The crack does so by healing, i.e., forming atomic bonds one by one, like closing a zipper. In reality crack healing is not often observed. This is not because the thermodynamics is wrong, but because surfaces are not flat to the atomic dimension, so that atoms cannot meet across the gap and form bonds. Several examples show that crack healing happens.
- *Adhesives*. Soft material can heal readily by flows.
- *Wafer bonding*. If the surfaces are indeed made flat, they will join.
- *Sintering*. At elevated temperatures, atoms can diffuse, so that the two surfaces change shape and can join.

Crack growth. If $a > a^*$, the elastic energy prevails over the surface energy. To reduce the free energy, the crack must grow. The crack does so by breaking atomic bonds one by one, like opening a zipper. This is the situation studied in this course.

The Griffith experiments. The main prediction of the Griffith theory can be written as

$$\sigma_c \sqrt{a} = \sqrt{\frac{2\gamma E}{\pi}} \ .$$

He performed several experiments to ascertain various parts of the equation.

Experiment 1. Confirm that $\sigma_c \sqrt{a} = \text{constant}$, independent of the size of the crack. Start with several glass sheets (large spherical bulbs actually). Introduce a crack in each sheet. Measure the strength of each sheet. Two important points:
(1) The crack introduced is in the mm to cm range, much longer than any "natural flaws" in the sheets, so that the natural flaws are negligible. In this way Griffith circumvented the uncertainties associated with the natural flaws.
(2) The introduced cracks in different sheets have different lengths, and the measured strengths are also different.

His data confirmed that $\sigma_c \sqrt{a} = \text{constant}$.

Experiment 2. Confirm that the constant is indeed $\sqrt{2\gamma E / \pi}$. Young's modulus for the glass used by Griffith was $E = 62$ GPa. The surface energy inferred from the measured breaking strength is $\gamma = 1.75$ J/m^2. Griffith needed an independent measurement of the surface energy. He did the creeping fiber experiment. The value he obtained was $\gamma = 0.54$ J/m^2. The agreement was fair.

Experiment 3. Measure strengths of glass fibers. For a fixed pre-existing crack size a, there is a critical stress:

$$\sigma_c = \sqrt{\frac{2\gamma E}{\pi a}} \ .$$

This is the stress needed to fracture the sample. This relation shows that the fracture strength depends on the crack size. Because different samples have different crack sizes, the fracture strength is not a material property. The measured strength has large scatter. Take representative values $\gamma = 1$ J/m^2, $E = 10^{11}$ N/m^2, $a = 10^{-6}$ m, the strength is 250 MPa. This corresponds to the experimental strength.

	Crack Length, $2a$ mm	Measured Strength, σ_c MPa	$\sigma_c \sqrt{a}$ MPa$\sqrt{m}$
sample 1	3.8	6.0	0.26
sample 2	6.9	4.3	0.25
sample 3	13.7	3.3	0.27
sample 4	22.6	2.5	0.27

(Data from the Griffith experiment)

Since the Griffith paper, the science of fracture was born.