Fracture of a steel. Following Griffith, you are performing the same experiment using steel rather than a glass. Let's say you have several bodies of the steel. Using a diamond saw, you cut each body with a crack of length $2a$. The lengths of the cracks are different in different bodies. You load each body in tension up to fracture, and record the applied stress at fracture σ_c . Many people have done such experiments and here are the basic experimental facts.

- $\sigma_c \sqrt{a} = \text{constant}$, independent of the length of the crack.
- The constant is orders of magnitude larger than $\sqrt{2\gamma E/\pi}$. Note that the surface energy of most solids is on the order of 1 J/m^2.

Thus, the Griffith theory agrees with one part of the experimental observation, but disagrees with the other. The large discrepancy between the Griffith theory and experiments with steels had to do with plastic deformation in the steel accompanying fracture. While other people complained about this large discrepancy, George Rankine Irwin (1907-1998) and Egon Orowan (1902-1989) did something about it: they invented a procedure to apply the Griffith theory to ductile materials such as steels.

Modify the Griffith theory to account for plasticity. Griffith's picture of fracture is

Fracture = atomic bond breaking.

Griffith used the surface energy to account for the inelastic process of bond breaking, and obtained the condition for fracture:

$$\sigma_c = \sqrt{\frac{2\gamma E}{\pi a}}.$$

Irwin's and Orowan's picture is

Fracture = atomic bond breaking + plastic deformation.

They define the fracture energy as the energy needed to advance a (steady state) crack by a unit area.

Fracture energy = surface energy + plastic work.

$$\Gamma = 2\gamma + w_p.$$

Here w_p is the work done to create per unit area of the plastic layers. Irwin and Orowan used the fracture energy to account for the inelastic process of bond breaking and plastic deformation, and they modified the condition for fracture as

$$\sigma_c = \sqrt{\frac{\Gamma E}{\pi a}}.$$

A few quick notes about fracture energy:

- The fracture energy is a material property, independent of the length of the pre-crack, so long as the small-scale yielding condition applies.
- The fracture energy is difficult to calculate from first principles, and is determined by fracture test, as described above.
- The fracture energy is much larger than the surface energy. A lot more atoms participate in plastic deformation than in bond breaking. Some rough values. Glass: 10 J/m^2. Ceramics: 50 J/m^2. Glassy polymers: 10^3 J/m^2. Aluminum: 10^4 J/m^2. Steel: 10^5 J/m^2.

The above modification eliminates the discrepancy between the theory and the experiments, but is bothersome in two respects. First, the Griffith theory was developed for a small crack in a large plate. How about other configurations of crack? Second, what do we really mean by the phrase "energy needed to advance a crack by a unit area"? We would like to have an operational definition of the fracture energy, a definition that will enable theoretical calculation and experimental measurement.

Small-scale yielding condition. We will be restricted to the case that the size of the plastic zone in the steady state is much smaller than the size of the crack, a condition known as the small-scale yielding condition. Under the small-scale yielding condition, much of the body deforms elastically. Because the size of the plastic zone is much smaller than the size of the crack, the crack can attain the steady state after extending by a length small compared to the total length of the crack.

The small-scale yielding condition involves the comparison of two lengths: the size of the plastic zone and the size of the crack. The size of the plastic zone in the steady state is a material property. For example, the plastic zone for silica is of atomic dimension, so that a crack beyond a few nanometers satisfies the small-scale yield condition. By contrast, the plastic zone for steel may be of millimeter in size, so that a crack beyond a few centimeters satisfies the small-scale yielding condition. For particularly ductile steel, however, the plastic zone can be several centimeters in size. To test such ductile steel under the small-scale yielding condition would require a body of a size about a file cabinet. Such a test is carried out sometimes, but is expensive.

Energy Release Rate. Let us first consider a pre-cracked body of an arbitrary shape. The body is purely elastic: no bond breaking or plastic deformation occurs. The body is loaded, say, by hanging a weight P. The elastic energy stored in the body U is a function of the displacement of the weight and the area A of the crack, namely,

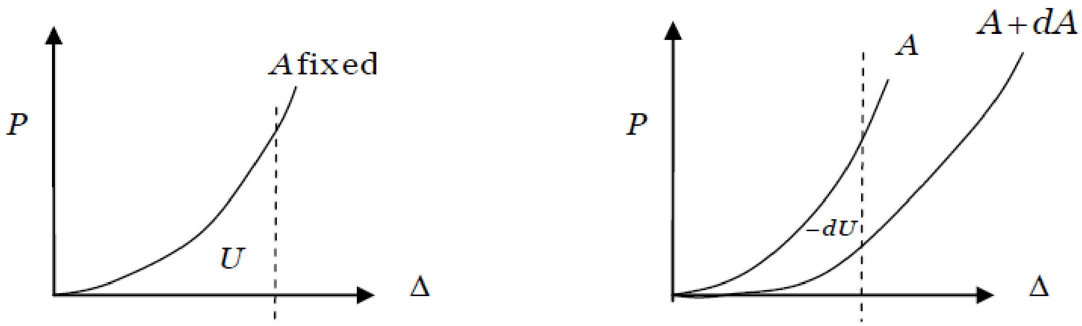
$$U = U(\Delta, A).$$

This function can be determined by solving boundary-value problems within the theory of elasticity.

Alternatively, the function can be determined by experimental measurement. For each copy of the body, we make sure that the crack is stationary as we load the body. Consequently, the work done by the weight is fully stored as the elastic energy in the body, $Pd\Delta = dU$. We write

$$P = \frac{\partial U(\Delta, A)}{\partial \Delta}.$$

By measuring the force *P* as a function of Δ and *A*, we can integrate and obtain the function $U(\Delta, A)$.



Define energy release rate, *G*, as the reduction of the elastic energy associated with the crack increasing per unit area, when the weight does no work, namely,

$$G = -\frac{\partial U(\Delta, A)}{\partial A}.$$

The partial derivative signifies that the displacement Δ is held fixed when the area of the crack *A* varies. Once we know the function $U(\Delta, A)$, the above definition gives the energy release rate *G*. Thus, *G* is a purely elastic quantity, and you need to know nothing about the process of fracture to obtain *G*.

When both the displacement of the weight and the area of the crack vary, the elastic energy of the body varies according to

$$dU = Pd\Delta - GdA.$$

Just as *P* is the thermodynamic force conjugate to the displacement Δ, the energy release rate *G* is the thermodynamic force conjugate to the area *A*.

Fracture energy. Consider a pre-cracked body loaded by a weight *P*. Under the small scale yielding condition, we can still obtain the function $U(\Delta, A)$ as if the entire body were purely elastic, either by solving a boundary-value problem with the theory of elasticity, or by the load-displacement curves determined experimentally with bodies containing cracks of different sizes.

When the weight drops by distance $d\Delta$, the weight does work $Pd\Delta$. Under the small-scale yielding condition, much of the work done by the weight is stored in the body as elastic energy, and only a small fraction of the work done by the weight goes to inelastic processes such as breaking atomic bonds and plastic deformation. We will use this small fraction to define the fracture energy. That is, the fracture energy is defined as an excess, according to

$$Pd\Delta = dU + \Gamma dA.$$

This definition of the fracture energy is independent of microscopic processes, be they bond breaking or plasticity.

Fracture criterion. Now compare the definitions of the energy release rate and the fracture energy. The crack will grow if the energy release rate equals the fracture energy:

$$G = \Gamma .$$

The energy release rate is the driving force for the extension of the crack. The fracture energy is the resistance to the extension of the crack. The relation between G and is analogous to the relation between stress and strength.

The above discussions complete the modifications of the Griffith theory to deal with

1. cracked bodies of any configuration, and
2. materials capable of inelastic deformation.

Below we collect a few useful mathematical refinements. These refinements often confuse students, but contain no new information.

Potential energy. View the body and the weight together as a system, and lump their energy together:

$$\Pi = U - P\Delta .$$

This quantity is called the potential energy in mechanics, and is called the Gibbs free energy in thermodynamics. This definition, in combination with $dU = Pd\Delta - GdA$, gives

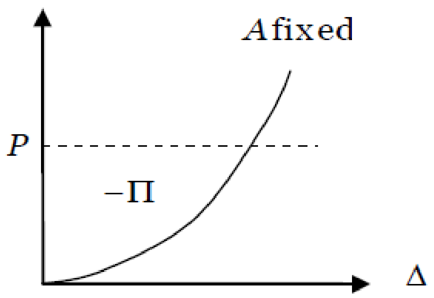
$$d\Pi = -\Delta dP - GdA .$$

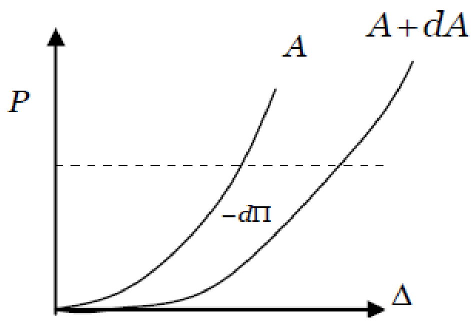
Now the potential energy is a function of the load and the crack area,

$$\Pi = \Pi(P, A) .$$

The displacement Δ and the energy release rate G are the differential coefficients, namely,

$$\Delta = -\frac{\partial \Pi(P,A)}{\partial P}$$
$$G = -\frac{\partial \Pi(P,A)}{\partial A} .$$





Linear elasticity. When the body is linearly elastic, the applied force P is linear in the displacement Δ. Consequently, the elastic energy is

$$U = P\Delta / 2 ,$$

and the potential energy is

$$\Pi = -U .$$

We can write the energy release rate as

$$G = +\frac{\partial U(P, A)}{\partial A} .$$

The partial derivative signifies that the load P is held fixed when the crack area A varies. The opposite signs in the two expressions for the energy release rate reflect a simple physical fact. When the area of the crack is larger, the body is more compliant, so that the body stores less elastic energy at a fixed displacement, but stores more elastic energy at a fixed load.

Compliance of a linearly elastic body containing a crack. For a linearly elastic body, the displacement is linear in the load. Write

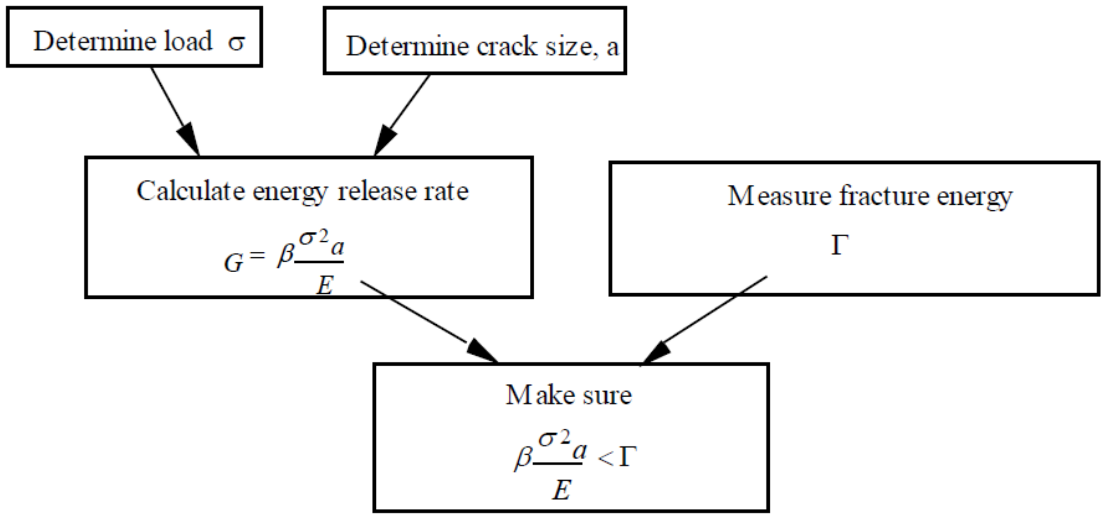
$$\Delta = CP ,$$

where C is the compliance of the body. For a linearly elastic body containing a crack, the compliance is independent of the load, but is a function of the area of the crack, namely, $C = C(A)$.This function can be determined experimentally or calculated by solving boundary-value problems. As we said before, the compliance is an increasing function of the area of the crack.

Using the compliance, we can write the energy release rate as

$$G = \frac{P^2}{2} \frac{dC(A)}{dA} .$$

Design based on fracture mechanics. Compare design based on fracture mechanics with design based on the linear elastic theory.



Ways to determine energy release rate. The energy release rate is a quantity defined within the theory of elasticity. The energy release rate is specific to the configuration of a cracked body, and can be determined by the following methods.

- Look it up in handbooks. Elasticity solutions to cracked bodies of many configurations can be found in handbooks, e.g., H. Tada, P.C. Paris and G.R. Irwin, *The Stress Analysis of Cracks Handbook*, Del Research, St. Louis, MO., 1995.

- Determine it experimentally. For a body containing a crack of a fixed area A, the displacement is linear in the force P, namely, $\Delta = CP$. The compliance C can be measured experimentally. Use several bodies, which are identical except for the areas of the cracks. Measure the compliance of each body, and obtain the function $C(A)$. The elastic energy stored in a body is $U=CP^2/2$. The energy release rate is given by

$$G = \frac{P^2}{2}\frac{dC(A)}{dA}.$$

- Determine it by solving the elasticity boundary-value problem. For cracked bodies of some configurations, the boundary-value problems can be solved analytically. For most configurations, the boundary-value problems are solved numerically by using finite element programs.

Historically, the analytical method came first, beginning with Griffith's (1921) use of the solution obtained by Inglis (1913), and followed by Obreimoff's (1930) analysis of a splitting layer. The method of functions of a complex variable was used to great effect by Muskhelishvili and others. The method of using the experimentally measured compliance to determine energy release rate was probably introduced by Irwin (~1950). The method is still occasionally used today.

For practical purposes, the method of choice today is often the finite element method.

Ways to determine fracture energy. Fracture energy is a material property. It can be determined in several ways.

- Look it up in a material data sheet. Representative values: Glass: 10 J/m^2. Ceramics: 50 J/m^2. Polymers: 10^3 J/m^2. Aluminum: 10^4 J/m^2. Steel: 10^5 J/m^2. Warning: The fracture energy is sensitive to the microstructure of materials; heat treatment of steel can change the fracture energy by orders of magnitude.
- Measure it experimentally by doing a fracture test. Of course, the values on the data sheet have been determined by experimental measurement.
- Compute it by a computer simulation of the fracture process. This is an emerging field. Exciting but immature. Not a standard engineering practice yet.

Applications of the fracture mechanics. The application of fracture mechanics is based on the equation

$$\Gamma = \beta \frac{\sigma_c^2 a}{E}.$$

Young's modulus is usually known. Of the other four quantities, if three are known, the equation predicts the fourth. If you just read this equation, fracture mechanics sounds like a silly tautology. It is not really so silly if you think through each application. Some quantities are easy to measure. Other quantities are easy to compute. One can make real predictions.

Application 1. Measure the fracture energy. Know β, σ_c, a. Determine Γ.

The experiment follows that of Griffith.

- Start with a body of a material.
- Cut a crack of a known size a using a saw.

- Load the sample with an increasing stress, and record the stress at fracture, σ_c.

- Separately find the elasticity solution for the energy release rate, $G = \beta\sigma_c^2 a / E$

- Convert the critical stress to the fracture energy, $\Gamma = \beta\sigma_c^2 a / E$.

The measured fracture energy is used to (a) rank materials, (b) study the effect of various parameters (e.g., loading rate, temperature, heat treatment) on fracture resistance, (c) design a structure to avoid fracture.

Application 2. Predict critical load. Know β, *a*, Γ Determine σ_c.

The body is given. The fracture energy of the material has been measured. The crack size *a* has been measured. Find the elasticity solution for the energy release rate $G = \beta\sigma_c^2 a / E$. This application requires one to determine the crack size. A large crack size is determined by visual inspection. A small crack can be determined by the x-ray or acoustic wave (Nondestructive Evaluation, or NDE). If the measurement technique cannot find any crack, simply put the smallest crack size can be detected by the technique (i.e., the resolution) into the equation, and predict a lower bound of the critical load. A crack in a structure may increase slowly over time. Inspect the structure periodically to monitor the crack size. Retire or repair the structure before the crack is too large.

Application 3. **Estimate flaw size from experimental strength**. Know β, σ_c, Γ. Determine *a*. This is the same as Griffith did.

- Measure the fracture load σ_c.

- Independently measure the fracture energy of the material.
- Approximate the energy release rate by that of a Griffith crack, $G = \pi\sigma^2 a / E$.
- The flaw size *a* is estimated by $a = \Gamma E / \pi\sigma^2$.

Application 4. Design a structure to avert fracture. If a material is given, and the load level is prescribed, one can design a structure to avoid fracture. One also needs to know the possible flaw size.