AHSS Formability – Sensitivity to Orientation

A solicited proposal (in response to solicitation dated January 23, 2009) to

The Auto/Steel Partnership
C/o Michael S. Bzdok
2000 Town Center, Suite 320
Southfield MI 48075-1123
Phone: 248-945-4778

by

Robert H. Wagoner, George R. Smith Professor
Department of Materials Science and Engineering
The Ohio State University
2041 N. College Road
Columbus, OH 43210
BACKGROUND

The A/SP solicitation to which this proposal responds is attached (Appendix A), along with other documents provided by A/SP along with the solicitation: A/SP Statement of Work (Appendix B), USAMP Purchase Order Terms and Conditions (Appendix C), and A/SP Deliverables and Guidelines (Appendix D).

Information contained in these documents will generally not be repeated in this proposal.

APPROACH

OSU will use the plane-strain tension test invented by the PI (References 1 and 2, attached as Appendices E and F, respectively) to accomplish the centerpieces of the project:
1) obtaining plane-strain load-displacement curves, and
2) measuring plane-strain formability (FLD_o) in RD and TD directions.

Some of the ancillary work will be provided by in-kind contributions as anticipated below. Because of the short deadline for this proposal, not every detail of the arrangement has been finalized for these anticipated in-kind contributions.

PLANE-STRAIN TENSION TEST

The plane-strain tension test devised by the PI in the 1970’s and in use since then will be used to produce the deformation states needed. It is described in some detail in References 1 and 2 (provided in Appendices E and F) along with applications extended to various materials and non-proportional tests (as outlined in References 3-7, not provided). The plane-strain tension test is ideally suited for the proposed work because it avoids the complexities of friction conditions and out-of-plane deformation.

Photographs of the test fixture with specimen and the grips available at OSU appear in Figure 1. A “B” type specimen, which is designed to accommodate large deformation of steel specimens, is shown. A full-size B specimen has a width in the reduced section of approximately 190mm.

The current load capacity of the load train at OSU is 150 kN. If necessary, this can be increased to 200kN with changes of the grips and load train pins. Even with this change, it will be necessary to scale the specimen, so the following proposal is based on redesigning the specimen to scale it to half scale in the plane (width 95mm), and to restrict the material choices to thicknesses smaller than 1.4mm for the strongest materials (UTS = 980 MPa). Thus, the highest expected load would be as follows:

Max. Load = 95 mm x 1.4 mm x 980 MPa x 2/sqrt(3) = 150 kN

All materials would be restricted to fall under this limit of thickness and UTS, i.e. 1.4mm x 980 MPa.
Figure 1 – Plane-strain tension testing: a) specimen and fixture installed in tensile testing machine, and b) exploded view of one grip for detail.
STATEMENT OF WORK

Task 1 – Plane Strain Testing (OSU)

  1. Redesign half-scale specimen, create CNC machining files.

  2. Machine 36 plane-strain specimens (18 RD, 18 TD) from 6 materials as listed in Appendix B ranging in thickness up to 1.8mm and UTS up to 980 MPa, but in combinations less than 1.4mm x 980MPa:

  3. Etch-grid front of each specimen for FLD measurement

  4. Measure load-elongation data for 24 specimens (12 RD, 12 TD), record

  5. Measure grid circles at incipient failure and adjacent to cracks, record strains

  6. Retain specimens for delivery to A/SP

Task 2 – FLDo Analysis (OSU)

  1. Read 0.1” (2.54mm) grid circles from each specimen adjacent to cracked regions.

  2. Plot strain combinations in standard FLD format, all of which are expected to be close to plane strain.

  3. Compute FLDo.

Task 3 – DIC Analysis (in-kind: Lou Hector, GM)

  1. Send specimens to GM / Lou Hector pre-deformation for paint spatter or other surface marking (rear surface) and for original DIC image photos*

  2. Send post-deformation DIC specimens to GM / Lou Hector for final DIC image photos and DIC analysis*

  3. Lou Hector to provide strain field maps from selected regions of 6 specimens.

* The details of how the DIC work will be accomplished may vary. Lou Hector, GM, has agreed to provide DIC work in-kind for this project. It may not be possible to work with photographs, in which case the specimens may be transported between GM and OSU. Or, it may be better to do the DIC work in real time, in which case either the grips and specimens would be transferred to GM for deformation of the specimens there, or the DIC equipment could be set up at OSU to record the deformation patterns.
Task 4 – OIM / EBSD Characterization (OSU)†

1. Strip off surface coating with acid. One specimen for each material.
2. Polish sheet surface to quality suitable for EBSD.
3. Scan with EBSD equipment – one scan normal to sheet surface for 6 materials
4. Analyze EBSD data for pole figure or color-texture plot.

DELIVERABLES

The OSU deliverables will be as stated in Appendix A, with the exception that in-kind deliverables will be prepared by the in-kind provider(s).

BUDGET

Budgeted duration of the project: Start: April 1, 2009, Finish: September 30, 2009

Budgets for Tasks 1 and 2 are shown separately from Task 3 in case an in-kind partner can be found for Task 3.

Budget - Tasks 1 and 2

<table>
<thead>
<tr>
<th>Item</th>
<th>Cost</th>
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<td>Graduate Research Associate (1/2 time)</td>
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<tr>
<td>CNC Machining</td>
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Budget – Task 4

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<tr>
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<tr>
<td>Fringe Benefits, Tuition:</td>
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<tr>
<td>Instrument Use Fees</td>
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</table>

† The OIM / EBSD Characterization is presented as to be done at OSU. If an in-kind partner can be identified for this part of the work, the budget items related to this task can be reallocated accordingly. The costs for OIM / EBSD are the least predictable in this proposal because surface preparation times to obtain a surface quality suitable for EBSD can vary greatly. Typical times are used for budgeting.
Indirect Costs $3,988
Total: $11,964

Project Total – Tasks 1-4 $78,209

REFERENCES


APPENDIX A: A/SP SOLICITATION
January 23, 2009

Dr. Robert H. Wagoner  
Department of Materials Science and Engineering  
The Ohio State University  
2041 College Road  
Columbus, OH 43210  
Phone: (614) 292-2079  
Email: wagoner@matsceng.ohio-state.edu

Subject: Request for Quotation

Dr. Wagoner:

The Advanced High Strength Steel Stamping Team of the Auto/Steel Partnership (A/SP) is requesting quotation for a project to measure AHSS formability sensitivity to orientation. The project is intended to experimentally test and measure plane strain forming limit (FLD₀) using simple plane strain tension test on the rolling direction, transverse direction and selected directions between the rolling and transverse directions.

The deliverables include:

1. Provide plane strain tested load-displacement curves for the measured materials.*  
2. Crystallographic orientation (pole figures or SEM-Electron Beam Selective Diffraction (EBSD)) and general optical micro-structural characterization (three directions) for each material tested. (Please quote as separate line item and reply if this deliverable can be provided as in-kind).  
3. Digital Image Correlation (DIC) Strain measurements. Please advise if this capability can be provided by OSU or if it needs to be supplied as in-kind by a member company.  
4. All raw data collected from tests, including thinning and circle grid reading following the common procedure of FLD₀ measurement.  
5. Description of test setup and specimen drawings with dimensions.  
6. Some typical samples after testing.  
7. Report describes:  
   • Test equipment and procedures.  
   • Data processing/calculation procedures and methodology.  
   • Comparison of formability in both directions.  

* Quality and extent of plane strain deformation within the sample to be agreed upon between the investigator and the A/SP Project Team
• All raw data sets in Microsoft Excel format.

A contract to the successful vendor will be issued by USAMP/DOE under recommendations by A/SP. The terms and conditions of the contract will be those of USAMP/DOE (attached). It is required that your proposal state specifically that your university accepts these terms and conditions and that you remove any of your university’s standard terms and conditions from your quote. The vendor shall submit a schedule for payment that includes the cost basis for the quotation.

The specific deliverables are:
1. A PowerPoint presentation of the results of the analysis (A short and a long version).
2. An Executive Summary of the analysis.
3. A written technical report of the analysis, detailing the findings, appropriate charts and conclusions. This will include the spreadsheet and the written instructions referenced in the deliverables.

All quotations shall be sent to the attention of Michael S. Bzdok, Project Manager at the Auto/Steel Partnership office, 2000 Town Center Drive, 3rd. floor, Suite 320, Southfield, Mi. 48075.

All responses to this request for quotation are due by COB, Monday, February 10, 2009

For technical questions on this solicitation contact:
Dr. Dajun Zhou (DJ)
Chrysler LLC
248-512-0906
djz13@chrysler.com

Please do not hesitate to contact me should you have any administrative questions.

Regards,

Michael S. Bzdok

Michael S. Bzdok
Project Manager
Auto/Steel Partnership
2000 Town Center, Suite 320
Southfield, MI 48075-1123
mbzdok@a-sp.org

O: 248-945-4778
F: 248-356-8511

Attachments
APPENDIX B: STATEMENT OF WORK
Statement of Work

Project
AHSS formability sensitivity on orientation

Project Lead
Dajun Zhou (DJ) Chrysler LLC

Project Description
It is noticed that some AHSS displayed a certain sensitivity of their formability to sheet rolling orientation in the cases of (A) small radius bending; (B) edge stretch; and (C) plane-strain stretch during tube hydro-bulging. This project is to experimentally test and measure plane strain forming limit (FLD0) using simple plane strain tension test on both the rolling direction and transverse direction.

Basic Test Condition
- The test should be well controlled without minor strain.
- The sample must be engraved with its material id and orientation.
- The test may be stopped at a certain percentage of load drops from the peak load to keep the deformed samples still in one piece.

Materials
- DP600 steel as a baseline material.
- One HSLA 50Ksi, three 780 UTS grade AHSSs (DP780, DP780HY and TRIP780) with the same strength level but different shapes of uniaxial tensile stress-strain curves and one DP980
- Thickness in the range of 1.2 to 1.8 mm
- Coated with GA, GI or EG
- Samples cut along the angle of 0, 30, 45, 60 and 90 degrees to the rolling direction

Deliverables
1. Provide plane strain tested load-displacement curves for the measured materials
2. Crystallographic orientation (pole figures or SEM-Electron Beam Selective Diffraction (EBSD)) and general optical microstructural characterization (three directions) for each material tested. (please quote as separate line item and reply if this deliverable can be provided as in-kind)
3. DIC Strain measurements
4. All raw data collected from tests, including thinning and circle grid reading following the common procedure of FLD0 measurement
5. Test setup and specimen dimensions.
6. Some typical samples after testing
7. Report describes:
   - Test equipment and procedures
   - Data processing/calculation procedures and methodology
   - Comparison of formability in both directions.
   - All raw data sets in Microsoft Excel format.
**Notes**
- How do we judge quality of plane strain deformation?
- In order to be a successful test the sample must fail at the center not the edge.
- The test should measure the reduction in thickness (cross sectional area) at fracture.
- The test needs to recognize the onset of localized necking.
- A digital imaging camera should be used to record the test data.
APPENDIX C: USAMP P.O. TERMS AND CONDITIONS
1. OFFER, ACCEPTANCE AND MODIFICATION – This order is an offer to Seller by Buyer to enter into the agreement it describes and it shall be the complete and exclusive statement of such agreement. Seller shall accept the offer in writing, or by beginning work hereunder of which Seller’s research administration authority, if any, has received actual notice. Modifications proposed by Seller are not part of the agreement in the absence of Buyer’s written acceptance.

2. CHANGES – At any time, the parties by written order may change the work of this order, including the specifications, statement of work, number and design of prototypes and delivery dates. If any such change affects cost or timing, the parties must adjust price and delivery schedules equitably. Seller may not make any change in the work of this order without the written approval of Buyer.

3. SUBCONTRACTING – Seller shall not subcontract any of its substantive obligations under this order without the prior consent of Buyer. In each subcontract of work hereunder, Seller shall obtain from the subcontractor the same obligations and rights and licenses for Buyer and Buyer’s partners and Partner Associated Companies as are provided by Seller under Section 9. A Partner Associated Company is any entity or division of a Partner, present or future, in which one of Buyer’s partners owns fifty percent or more of its voting stock or equity.

4. TITLE AND BAILMENT PROPERTY – (a) Any documents and articles produced or acquired by Seller under this order, but not (i) intellectual property rights thereto or (ii) material and equipment under $2,000 in value, become the property of Buyer immediately upon production or acquisition. (b) Unless otherwise specified, Seller bears all responsibility for loss and damage to all documents and articles owned by Buyer and possessed by Seller, excluding normal wear and tear. Seller shall (1) properly house and maintain such documents and articles on Seller’s premises, (2) mark them “Property of 10AMP”, (3) retain in nonincriminating files with the accounts of Seller or with that of a third party, and (4) maintain them as personalty. Buyer may take, at Seller’s expense of, a reasonably thorough inventory of such documents and articles at any reasonable times to inspect the documents and articles and pertinent records. After completing the work of this order, Seller shall advise Buyer of those documents and articles produced or acquired under this order that remain in Seller’s possession. (c) At Buyer’s request, Seller shall immediately deliver the documents and articles to Buyer or to a carrier selected by Buyer, of Buyer’s option F.O.B. carrier’s facility or F.O.B. Buyer’s facility freight collect, properly packed and marked in accordance with the requirements of the carrier and Buyer. If Buyer requests Seller to scrap the documents or articles, Seller must destroy the specified documents and articles or mutilate them to the point of uselessness only as raw materials. Seller may sell the materials resulting from such mutilation only to another who agrees to use them only as raw materials. Seller may delegate to a responsible third party its duties regarding the destruction or mutilation of such documents and articles, but delegation does not relieve Seller from responsibility for such duties and Seller must monitor the performance of the third party.

5. BLANKET ORDER RELEASES – If this purchase order specifies that the services to be performed shall be designated by release, Seller shall perform services only as authorized in releases issued to Seller by Buyer. Any specific requirements concerning scheduled milestones, delivery dates or progress reporting must be met by Seller prior to payment by Buyer, including progress payments.

6. INVOICES AND PAYMENT – (a) “Costs” means allowable costs incurred by or on behalf of Seller in performing this order in accordance with the provisions of the Federal Acquisition Regulation (FAR) at 48 CFR part 31.2 and 10 CFR part 600.127. Unless applicable government regulations apply other rules, indirect Costs will be determined in accordance with generally acceptable accounting principles consistently applied. Seller will invoice Buyer monthly for all Costs, and will submit a final invoice within 90 days of completion of the work under this order. Each invoice must specify the full amount for direct labor, direct materials, indirect costs and other appropriate data requested by Buyer, and be submitted in a form acceptable to Buyer. Buyer’s payment obligation is based on the allowable and allocable Costs incurred by Seller in performing the work required by this order. Seller shall provide written notice to Buyer when work performed reaches eighty percent of the maximum price, if any, specified in this purchase order. If applicable to Seller, each invoice shall contain the following assurance: “Seller represents that it has complied with the Fair Labor Standards Act of 1938, as amended, in producing the supplies or performing the services covered by this invoice.” (b) Buyer intends to capture as in-kind contribution for the United States Department of Energy (DOE) Cooperative Agreement DEFC05-95OR22363 all allowable and allocable Costs incurred by or on behalf of Seller under this order. Costs contributed by Seller may not be charged to Buyer or the Government under any other grant, cooperative agreement, or contract. (c) Seller shall establish an accounting system that enables ready identification of the foregoing data, including identification of Costs, and maintain records of all contract costs incurred and invoiced under this order for 3 years after final payment under this order. If an audit, litigation or other action involving the records is started before the end of the 3-year period, the records must be retained until all issues arising out of the action are resolved. Buyer, DOE, the Comptroller General of the United States, or any of their authorized representatives, as far as the applicable records are retained by Seller, shall have the right of access to any books, documents, papers or other records of Contractor which are directly pertinent to this order, in order to make audit, examination, excerpts and transcripts. Seller shall ensure that its subcontractors meet the applicable record and audit requirements of 10 CFR part 600.

7. PERFORMANCE – Seller expressly represents that all goods or services covered by this order will conform to the specifications furnished to or by Buyer.

8. INFRINGEMENT – On reasonable terms and conditions, Seller must cooperate with Buyer in Buyer’s investigation, defense or handling of any claim that a third party may bring against Buyer, its partners, or Partner Associated Companies, or others that use the documents and articles on behalf of any of them, for any alleged infringement of any present or future patent, copyright, industrial design right or other proprietary right based on Seller’s work hereunder or the sale or use of the documents or articles (1) alone, (2) in combination by reason of their content, design or structure, or (3) in combination in accordance with Seller’s recommendations.

9. INFORMATION AND DATA – (a) Seller, without restrictions of use or disclosure, must furnish to Buyer or another party designated by Buyer all information and data Seller acquires or develops in the course of Seller’s activities hereunder. At Buyer’s request, on reasonable terms and conditions, Seller must disclose to Buyer or another party by Buyer, any potential design, quality or manufacturing problems with articles Seller worked on or produced pursuant to this order. (b) At Buyer’s request, on reasonable terms and conditions, Seller must furnish to Buyer, all other information and data of Seller that Buyer considers necessary to understand and apply the information and data of section 9(a).

10. ENGINEERING DRAWINGS – Any engineering drawings Seller is required to prepare and furnish to Buyer shall conform with standards to be provided by Buyer.

11. INDEMNITY – To the extent permitted by the laws of the state in which the Seller is organized, and not inconsistent with the doctrine of sovereign immunity, Seller accepts full responsibility for any loss or personal injury that may result from negligence of Seller or any employee, agent, or duly authorized representative of Seller.

12. TERMINATION AT OPTION OF BUYER – (a) Buyer may terminate its purchase obligations hereunder, in whole or in part, at any time, by a written notice of termination to Seller. Buyer shall have such right of termination notwithstanding the existence of an excusable delay of Section 14. (b) Upon receipt of the notice of termination, Seller, unless otherwise directed by Buyer shall (1) terminate promptly all work under this order, (2) transfer title to and deliver to Buyer the finished work, the work in process and the parts and materials which Seller produced or acquired in accordance with this purchase order and which Seller cannot use in producing goods for itself or for others, (3) settle all claims by subcontractors (if any) for actual costs that are rendered unrecoverable by such termination, and (4) take actions reasonably necessary to protect property in Seller’s possession in which Buyer has an interest. (c) Upon termination by Buyer under Section 12, Seller’s obligation to Seller shall be: (1) the purchase order price for all finished work and completed services which conform to the requirements of the order, (2) Seller’s reasonable actual cost of the work in process and parts and materials transferred to Buyer in accordance with subsection (b)(2) hereof, (3) Seller’s reasonable actual cost of settling the claims by subcontractors of subsection (b)(3) hereof but not in excess of the obligation; Seller would have had to the subcontractor in the absence of termination, and (4) Seller’s reasonable actual act of carrying out its obligations of subsection (b)(4) hereof. Seller’s obligations upon termination under this Section shall not exceed the obligation Buyer would have had to Seller in the absence of termination. (d) Within two months after the date of termination, Seller shall furnish to Buyer its termination claim which shall consist exclusively of the items of Buyer’s obligation to Seller that are listed in subsection (c) hereof. Buyer may audit Seller’s records, before or after payment, to verify amounts requested in Seller’s termination claim. (e) Buyer shall have no obligation to Seller if Buyer terminates its purchase obligations of the purchase order because of default by Seller.

13. COMPLIANCE WITH LAW – Seller shall comply with federal, state and local laws, rules, regulations, ordinances and executive orders applicable to Seller’s performance of its obligations under this order. Contract clauses required by the Government in such circumstances are incorporated herein by reference.

14. EXCUSABLE DELAYS – Neither Buyer nor Seller shall be liable for a failure to perform that arises from causes or events beyond its reasonable control and without its fault or negligence, including labor disputes of any kind. Seller’s delivery obligations of Section 4 are not impaired by an excusable delay of this Section.
APPENDIX D: A/SP – DELIVERABLES GUIDELINES
Auto/Steel Partnership – Deliverables

Guidelines

Please thoroughly read the following instructions regarding the preparation of the project deliverables (project results, engineering reports, technical articles, etc.).

PLEASE DO NOT INCORPORATE VENDOR LOGOs within the technical reports. The A/SP logo is the only logo that should be included on the final deliverables.

Formatting
Use Microsoft Word, Word Perfect, PageMaker or Quark XPress
1" margins around entire document

Title of Paper
Bold/Upper & Lower Case
Capitalize all words except prepositions and incidentals

Body Copy
10pt Book Antigua font
Author & Affiliation: Italic/Upper & Lower Case

Headings within body of text (Such as, Abstract, Introduction, Conclusion, References, etc.)
Bold/Upper & Lower Case and capitalize all words except prepositions and incidentals.

Figures Captions
Bold/Italic
Spell out and capitalize the word "Figure" throughout manuscript. Please note there is no punctuation at the end of captions except for question marks and parenthesis.

Tables Captions
Bold/Italic and capitalize the word "Table" throughout manuscript. Please note there is no punctuation at the end of captions.

Outlined Information
Regular/Capitalize the first work of each line.
Follow the example for preparing outlined information:
• Bullet item
- Hyphenated item
1. Numbered item
  A. Lettered item if special scientific symbols are not available, they may be inserted by using black ink.
  (Do not use any undefined acronyms)

**Equations**
Please note equation number is in parenthesis and right justified.

**Graphics**
Graphics should be placed inside the final report and saved individually on CD ROM.

**References in Body of Paper:**
References are to be in brackets, [7], while multiple references are, [4,8,10]

**References (End Notes)**
Numbered 1., 2., etc. - Regular & Italic/Upper & Lower Case/Numbered, 10 point text, for example:
2. Mark A. Jenkins, Philip Woods and Kuniko Urba, "Reduction in NOx from Combustion Fuel Gases."

**Digital Guidelines for Final Reports**
The final project results should be saved in its original format with graphics placed in the body of the paper. This file must not be compressed.

Photographs and other scanned art must be saved as TIFFs or JPEGs. Do not apply LWZ compression to these files. All scans and/or digital photography must be at 300dpi, at the desired final size. All bit mapped scans (black & white bmps) should be at least 600dpi, at the final size. All other types of graphics should be saved as .eps documents or converted to TIFFs or JPEGs. Authors must include hard copy of their graphic files along with all original photographs and artwork.

All files except for the text file can be compressed (zipped) by only the following programs: WinZIP, Stuff It, Stuff It Deluxe, Compact Pro, Disk Doubler or Zip. Do not use compression programs that results in an ".arj" file extension or in multiple file segments.

Media Files can be stored on CD-R/RWs, DVD-R/DVD-RW (+) (-)

Provide hard copies of all tables, figure illustrations, and charts. All camera-ready graphics including line illustrations, charts, graphs and tables must be of the highest quality and suitable for reproduction. All graphics must have a figure caption. Please print the figure number and the last name of the author on the back of each original. Also, mark with an arrow to show the top of the graphic. Please use a pencil for all markups.

Additional templates can be found on the enclosed CD ROM, including:
• PowerPoint Presentation Template
• CD Jewel Case Cover
• CD Label
• Fonts for Templates

Thank you for your cooperation
APPENDIX E:

AN EXPERIMENTAL AND ANALYTICAL INVESTIGATION OF IN-PLANE DEFORMATION OF 2036-T4 ALUMINUM SHEET

ROBERT H. WAGONER and NENG-MING WANG
General Motors Research Laboratories, Warren, MI 48090, U.S.A.

(Received 26 September 1978; in revised form 20 January 1979)

Abstract—Sheet aluminum alloy (2036-T4) specimens of several geometries were photogrided and pulled in a tensile testing machine while precision photographs were taken of the photogrid. This technique allowed determination of strain distributions and load-displacement points. These results are compared with corresponding results obtained by Finite Element Modeling based on Hill's anisotropic plasticity theory and experimental tensile stress–strain data. FEM predictions and experimental results are in excellent agreement; verifying Hill's model for the case of in-plane deformation of 2036-T4 aluminum alloy between the strain states of plane strain tension and uniaxial tension.

NOTATION

$A$ angle of notch side in the sheet metal specimens, degrees (Fig. 1)

$B$ radius of curvature in the notch of the sheet metal specimens, mm (Fig. 1)

$C$ gage length in the sheet metal specimens, mm (Fig. 1)

$r$ plastic anisotropy parameter, defined as the ratio of the width strain to thickness strain in a uniaxial tension test

$\Delta r$ planar anisotropy parameter

$\epsilon_l$ logarithmic in-plane principal strains

$\epsilon$ effective (or equivalent, or generalized) strain, as defined by the Hill normal anisotropic plasticity theory

$\varrho$ effective stress, corresponding to the effective strain

$\sigma_0$ saturation stress in the saturation stress hardening model, MPa

$a$ preexponential coefficient in the saturation stress hardening model

$b$ exponential argument coefficient in the saturation stress hardening model

$k$ strength coefficient in the power law hardening model, MPa

$E$ Young's modulus, GPa

$n$ work hardening rate in the power law hardening model

$\sigma_0$ prestrain term (when used) in the power law hardening model

$X_{i0}, X_{j0}$ deformed and initial Cartesian coordinates in the $i$th direction of the $j$th node of an element

$M_{ij}$ elementary stiffness matrix elements

$U_{ij}$ displacement component of the $j$th node of an element in the $X_{i}$th direction

$t_{ij}$ force component of the $j$th node of an element in the $X_{i}$th direction

$D$ imposed displacement boundary condition in the FEM program

$t$ a dot above a variable indicates a time derivative.

INTRODUCTION

Theories of anisotropic plasticity based on modifications of the von Mises[1] yield criterion and its associated flow rule were developed independently by Jackson et al.[2], Hill[3, 4], and Dorn[5] beginning in 1939. The most generally quoted, because of its rigor and simplicity, is that of Hill. The anisotropy considered is orthotropic, i.e. possessing three mutually orthogonal symmetry planes, with normal or planar anisotropy being special cases. The plastic properties are considered uniform and homogeneous and no Bauschinger effect nor hydrostatic component of stress effect are included. Isotropic hardening is implicit in the model, which in turn implies no changes in the plastic anisotropy during deformation.
Although plasticity theory has been extensively used to model special types of
technologically important forming processes[6] such as deep drawing, bore expanding,
and stretching, direct comparisons between theoretical solutions and experiment are
rare.

Richmond et al.[7] found good qualitative agreement between strain distributions
in uniaxially stressed rectangular sheets and finite-element method (FEM) predictions
for several steels. They also found reasonable agreement in load-displacement curves
for uniaxially loaded notched bars. Wang and Budiansky[8] compared strain dis-
tributions with predictions for stretching over a hemispherical punch but had to use an
arbitrary coefficient of friction to optimize the agreement.

It should be mentioned that most tests of plasticity theory have taken place in
balanced biaxial tension, as is obtained in a hydraulic bulge test[10–15]. The results of
Pearce[13] and Woodthorpe and Pearce[14] on a large variety of sheet metals show
that Hill’s theory is generally unsatisfactory for materials with η value less than unity,
although we find their presentation of the comparisons confusing, see Discussion. The
only reported tests in plane strain compression appear to be those of Taghvaipour and
Mellor[16] on steel, aluminum, titanium and zinc sheet. They found reasonable
correlations to Hill’s theory for steel and aluminum but little agreement in the case of
titanium and zinc.

The purpose of this work was to assess the accuracy of Finite Element Modeling
(based on Hill’s theory) in predicting strains and loads during in-plane deformation of
2036–T4 aluminum alloy sheet. This was done by obtaining experimental data of
in-plane deformation of sheet specimens of several geometries and comparing them
with the corresponding analytical results based on Hill’s theory. This aluminum alloy
was chosen because of its application in the automobile industry and also because two
of its material characteristics: negligible strain rate sensitivity[9] and little planar
anisotropy, closely meet the material assumptions already implemented in an existing
FEM program[8] and the assumptions upon which Hill’s theory is based. The
geometries of the sheet specimen are such that the strain states are bounded by
uniaxial tension and plane strain. Thus, the conclusions drawn from this work should
be viewed as valid only in this range of strain states.

**PROCEDURE**

*Tensile tests*

Tensile tests were performed in the rolling, transverse, and 45° direction using ASTM E-8 tensile
specimens with 50.8 mm gage lengths. Each specimen was photogrided with 2.54 mm dia. circles and
pulled to failure at a crosshead rate of 8 x 10^-4 mm/sec. Photographs of the grid were taken at one minute
intervals and the loads were simultaneously recorded. Strains were measured over a 7.5 mm gage length
(three circles) nearest the eventual fracture location. This procedure allowed direct r-value determination
and measurement of stress–strain data somewhat beyond uniform elongation. It should be noted that the
true strains cannot be measured accurately when localization occurs over a distance smaller than 7.5 mm.
The average overall strain rate in these tests was 10^-7/sec.

*Shear specimen tests*

The shear specimen tests were performed in much the same way as the tensile tests described above.
Five types of specimens (Fig. 1) were machined from the same 2036–T4 aluminum sheet as the tensile
specimens with the rolling direction parallel to the tensile axis. In one case (specimen B) a specimen was
also machined with the transverse direction parallel to the tensile axis. All specimens were then photogrid-
ded with 2.54 mm circles and mounted in the tensile testing machine as shown in Fig. 2.

The specimen gripping arrangement is illustrated in Fig. 3. These grips originally met the specimen
throughout the serrated grip faces but this was found to allow significant distortion within the grip region,
thereby producing unknown boundary conditions. The grip faces were relieved in a band across the center
of the specimen to increase the pressure at the meeting line closest to the center of the specimen. This reduced
slippage and assured maintenance of a line of very little distortion at the grip-specimen interface. This
boundary condition is amenable to finite element modeling. The bolts through the grips were torqued to
110 N·m.

The crosshead displacement rate used in these tests was 8 x 10^-3 mm/sec, equivalent to average overall
strain rates of 0.9 x 10^-3/sec in the center of the specimen and 1.3 x 10^-3/sec at the edges, along the
centerline. Pictures were taken at 1 min intervals and loads were recorded simultaneously.

X and Y strains were measured at fourteen intervals across the centerline of the specimen using a
traveling microscope. In each case, three circles were used as a gage length to increase the accuracy of the
method. The strains were determined by dividing the three circle length by the initial three circle length
corresponding to that position and taking the natural logarithm. Several tests were performed with a
Investigation of in-plane deformation of 2036-T4 aluminum sheet

![Diagram of specimen geometry with dimensions]

<table>
<thead>
<tr>
<th>Specimen</th>
<th>Thickness (mm)</th>
<th>A (degrees)</th>
<th>B (mm)</th>
<th>C (mm)</th>
</tr>
</thead>
<tbody>
<tr>
<td>A</td>
<td>1.252</td>
<td>0</td>
<td>19.1</td>
<td>12.7</td>
</tr>
<tr>
<td>B</td>
<td>1.255</td>
<td>45</td>
<td>19.1</td>
<td>12.7</td>
</tr>
<tr>
<td>B (transverse)</td>
<td>1.262</td>
<td>45</td>
<td>19.1</td>
<td>12.7</td>
</tr>
<tr>
<td>C</td>
<td>1.245</td>
<td>30</td>
<td>19.1</td>
<td>12.7</td>
</tr>
<tr>
<td>D</td>
<td>1.300</td>
<td>45</td>
<td>12.7</td>
<td>12.7</td>
</tr>
<tr>
<td>E</td>
<td>1.252</td>
<td>45</td>
<td>25.4</td>
<td>0.0</td>
</tr>
</tbody>
</table>

**Fig. 1.** General sheet specimen geometry with dimensions for each specimen.

![Diagram of cross section of a tensile grip]

**Fig. 3.** Cross section of a tensile grip showing the clamping mechanism and grip face relief.

standard, undeformed grid mounted so as to be photographed with the deforming one. In this way, it could be determined whether the camera magnification drifted during a test. In no case was drift found outside the experimental uncertainty. The estimated accuracy of the strain readings is 0.003.

The axial displacements of the specimen bounding lines (dashed lines, Fig. 1) were also determined from the photographs. Y-strains were determined using the largest gage length available in the center of the specimen and these were then converted to displacements based on a stationary transverse center line. That is, the gage length for the presented displacements is 45 mm—half the gage length that would be used in a grip-to-grip displacement measuring procedure.

**Finite element analysis**

Finite element calculations for the sheet specimens were carried out using an existing large displacement-large strain finite element program developed for sheet metal stamping operations. Since a detailed
description of the finite element program has already been published elsewhere[8], we list here only its major assumptions and principal features as follows:

1. The sheet material is assumed to be normally anisotropic and satisfies Hill's theory for anisotropic materials;
2. The stress and strain states in the sheet are prescribed by the membrane theory;
3. For the material's constitutive relations, the Jaumann rate of Kirchoff stress is assumed to be a linear function of Lagrangian strain rate;
4. Constant stress triangular elements are used to represent the initial and deformed sheet domain;
5. A step-by-step incremental scheme is used to account for both geometric and material nonlinearities.

In the original program, both in-plane and out-of-plane sheet metal displacements were taken into account, as were the constraints imposed by punch and die constraints. For the sheet specimens considered in this work, only in-plane deformations were involved. Consequently, for a typical triangular element as shown in Fig. 4, the elementary stiffness equations are simply:

\[
\begin{align*}
M_{ij} \dot{U}_i &= \dot{t}_j \\
\end{align*}
\]

where \( \dot{U} \) and \( \dot{t} \) are the in-plane nodal velocity and force rate vectors respectively, and \( M_{ij} \) is the in-plane stiffness matrix. The vectors \( \dot{U} \) and \( \dot{t} \) and the matrix \( M_{ij} \) have been defined respectively by equations (28), (35) and (39) of Ref. [8].

As a demonstrative example, we show in Fig. 5 a finite element model of specimen B. Because of the specimen symmetry (two mirror planes), only one quarter of the specimen needed to be analyzed. The boundary conditions are such that symmetry conditions are enforced along the left and lower edges (in Fig. 5) and a rigid body motion given by

\[
\begin{align*}
\dot{U}_1 &= 0 \\
\dot{U}_2 &= D
\end{align*}
\]

is satisfied on the upper edge which physically corresponds to the clamped end. With the displacement \( D \) of the clamped edge as the monotone quantity of the stretching process, the stresses and strains in the specimen were calculated by repeatedly updating and assembling equation (1) and solving the resultant master stiffness equations. The loads required to displace the top edge of the specimens were calculated by summing the nodal forces acting on the upper (or lower) edge. The step size used in all calculations was: \( \Delta D = 0.0125 \text{ mm} \).
Fig. 2. Photograph of a photogridded sheet specimen mounted in the tensile grips.
Table 1. Summary of tensile test data for 2036-T4 aluminum. All curves are fit to stress-strain data in the strain range 0.01-0.18

<table>
<thead>
<tr>
<th>Tensile axis</th>
<th>( \sigma = Ke^\epsilon )</th>
<th>( \sigma = K(\epsilon + e_0)^n )</th>
<th>( \sigma = \sigma_0(1 - a \exp(-b\epsilon)) )</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>( r )</td>
<td>( K ) (MPa)</td>
<td>( n )</td>
</tr>
<tr>
<td>T. D.</td>
<td>0.676</td>
<td>593</td>
<td>0.222</td>
</tr>
<tr>
<td>R. D.</td>
<td>0.701</td>
<td>595</td>
<td>0.213</td>
</tr>
<tr>
<td>45°</td>
<td>0.752</td>
<td>583</td>
<td>0.214</td>
</tr>
<tr>
<td>Average</td>
<td>0.707</td>
<td>589</td>
<td>0.216</td>
</tr>
</tbody>
</table>

\( \dagger S \) = standard deviation of the fit = \( \sqrt{\frac{\sum (\sigma - F(\epsilon))^2}{N - f}} \) where \( F \) is the function being fit and \( f \) = number of fit parameters.

\( \ddagger \) All plastic properties are averaged by the rule:

\[ X = \frac{X_a + X_b + 2X_c}{4} \]

RESULTS

A summary of the tensile test data is presented in Table 1. As the table shows, this alloy exhibits little planar anisotropy (\( \Delta r = -0.037 \), \( \approx 5\% \) of \( r \)). A saturation model hardening law fits the tensile data very well; much better than a power law-type law. The averaged values of \( \sigma_0, a, b \) were used in conjunction with the saturation stress model in the FEM program.

Experimental and theoretical strain distributions at several displacements are presented for the sheet specimens in Fig. 6. In each case the experimental strains presented are taken from the photographs taken at the grip displacement closest to the one at which the theoretical strain distribution was determined. The agreement in all cases is striking. The only differences occur at the outer edges, where in some cases the experimental strains tend to be low. This effect is probably a result of minor slippage in the gripped region near the edge of the specimen.

The experimental and theoretical load-displacement plots are presented in Fig. 7. Once again the agreement is very good, with any differences probably attributable to machine calibration and variation in specimen thickness.

In order to check the assumption of planar isotropy, a B-type specimen was machined with the transverse direction parallel to the tensile axis. The strain distribution comparisons for this specimen are presented in Fig. 6(f) and the load-displacement comparison is presented in Fig. 7(f). In both cases, the agreement is as good as in the rolling direction specimens, indicating a high degree of isotropy in the sheet plane.

Qualitatively, fracture of the sheet specimens originated in the notch radius region, in agreement with the maximum strain loci found by FEM. Specimen E broke near the centerline. Based on a maximum strain criterion, the FEM results predict approximately the same displacement-to-failure for specimens A, B and C; a somewhat lower limiting displacement for specimen D, and the lowest of all for specimen E. This is the observed ranking, as can be seen in Fig. 7.

DISCUSSION

We have limited this paper to a quantitative comparison of experimental strain distributions and deformation loads with ones derived from finite element modeling based on Hill’s theory and tensile test results. The agreement in all cases is too good to be fortuitous, but some care should be taken in interpreting these results. The limitations are as follows:

1. The only strain states encountered in the deformation of these specimens lie between plane strain tension and uniaxial tension. Care should be taken in extrapolating to other strain states. Note particularly Hill’s new theory\(^d\) diverges most as the strain state approaches balance biaxial tension.

2. The aluminum alloy chosen for these tests meets the assumptions inherent in Hill’s normal anisotropy plasticity theory, namely: little strain rate sensitivity, little planar anisotropy, and little change in normal anisotropy with progressive straining.

3. The specimens used here produce nearly proportional straining for each material element such that path-dependent effects are probably not reflected.

4. The strain range encountered along the centerline of these specimens is limited to approx. 0.15% although other elements in the specimen are strained to failure.

An apparent conflict of the present research with existing literature occurs by extrapolating our results to the case of balanced biaxial tension, which has been found anomalous by Woodthorpe and Pearce\(^{[14]}\). There appears, however, to be some ambiguity in their method of comparison. Their equation (4) is used for predicting equivalent biaxial strains from measured uniaxial ones and is applicable to through-thickness strains when the balanced biaxial directions lie in the sheet plane. Their measured strains, however, are taken from circles at the pole of the bulge, such that they measure in-plane strains, different by a factor of two from the strains for which equation (4) applies. If their Fig. 1 is replotted with this correction, the uniaxial and biaxial curves are in better agreement.

Despite the above conjecture, it may be that the largest divergences from Hill’s old theory occur near balanced biaxial tension and do not show up in our strain state range. Commercially pure aluminum, tested by Woodthorpe and Pearce and 2036-T4 aluminum alloy may have significantly different plastic properties away from uniaxial tension.
Fig. 6. Comparison of experimental strain distributions measured at the transverse center line of sheet specimens with FEM-generated strain distributions at two displacements.
CONCLUSIONS

(1) The technique of continuous in-plane deformation of arbitrarily shaped sheet specimens with photographically recorded strains has been found to be an accurate experimental method of observing deformation under various strain states.

(2) Strain distributions and loads can be accurately predicted using FEM and tensile stress–strain data for 2036–T4 aluminum alloy during in-plane deformation.
(3) Hill's normal anisotropic plasticity theory is completely satisfactory for use with in-plane deformation of 2036-T4 aluminum between the strain states of plane strain tension and uniaxial tension.

Acknowledgements—We would like to thank Mr. Paul Frechette for his painstaking experimental work and attention to detail necessary in determining the strain distributions presented here and Dr. William Frey for making available a finite element mesh generating procedure.

REFERENCES

APPENDIX F:

Measurement and Analysis of Plane-Strain Work Hardening

R. H. WAGONER

A new technique for measurement of plane-strain work hardening has been developed which uses tensile loading and computer analysis for interpretation, and which eliminates the experimental uncertainties of large strain gradients, friction, and out-of-plane bending inherent in the usual plane-strain deformation modes. Plane-strain and tensile work-hardening curves have been measured for 2036-T4 aluminum alloy using several types of sheet specimens. The work-hardening rate in plane strain is lower than that in uniaxial tension. In each case a Voce-type empirical work hardening law represents the data well. Hill's theories cannot account for these data because the isotropic hardening assumption is violated. A method of analysis was introduced to determine Hill's new m parameter as a function of strain and m was found to vary from 1.6 to 2.0 in the strain range 0.02 \( \leq \varepsilon \leq 0.18. \)

Predictive calculations of press loads, strain distributions, and failure loci in sheet metal forming operations depend on an accurate knowledge of plastic behavior under various strain states. For these reasons, the development of a general yield criterion and plasticity theory has been pursued for many years. The purpose of the present work is to experimentally and analytically examine plastic behavior under one of the most widely encountered (and often most critical) strain states in sheet stamping operations: plane strain.

Jackson et al., Hill, and Dorn independently developed theories of anisotropic plasticity based on modifications of the von Mises yield criterion and its associated flow rule. Hill's original theory (let us call this Hill's 'old' theory) is the most widely used today. The anisotropy is assumed to be orthotropic, i.e., possessing three mutually orthogonal symmetry planes, with normal or planar anisotropy being special cases. The plastic properties are considered uniform and homogeneous and neither Bauschinger nor hydrostatic stress effects are included. Isotropic hardening is also assumed; in turn implying that the plastic properties do not depend on the strain state or stress.

When only normal anisotropy is considered; that is, all directions in the sheet plane are equivalent, the only parameter to be measured for application of the theory is the average plastic anisotropy ratio, \( \tilde{\tau} \). Once this parameter is determined from tensile tests the theoretical predictions are fixed; that is, there are no other adjustable parameters.

Hill's 'old' theory has recently been tested using various sheet alloys. The most common test is a comparison of stress-strain curves obtained in balanced biaxial tension (as in a hydraulic bulge test) with curves predicted using the theory based on tensile test data. A similar analysis has been applied to experiments carried out in plane-strain compression.

The conclusions reached through such tests have been mixed. The theory has been found to predict the behavior of aluminum-killed steels \( \tilde{\tau} = 1.3 \) to 1.6 but has not been satisfactory for rimmed steels \( \tilde{\tau} = 0.4 \). Results for titanium sheet have also varied. Pearce found good agreement between balanced biaxial experiments and theory for Ti 115 (\( \tilde{\tau} = 3.8 \)) whereas Bramley and Mellor and Tagghvaipour and Mellor found marginal agreement in balanced biaxial tension and plane-strain compression for titanium sheet (\( \tilde{\tau} = 2.9 \) to 4.2).

Tests of Hill's 'old' theory for other alloys have shown poor agreement. Zinc \( \tilde{\tau} = 0. 2 \) to 0.4, copper \( \tilde{\tau} = 0.8 \) and 70/30 brass \( \tilde{\tau} = 0.8 \) sheets all exhibit large deviations from predictions. Rogers and Roberts have interpreted the failure of the theory in terms of crystallography and Dillamore concluded that the only agreement likely to be found between theory and experiment would occur in materials with \( 1 \leq \tilde{\tau} \leq 2. \)

The plastic response of aluminum sheet alloys has been investigated in uniaxial tension, \( \tilde{\tau} = 0.8 \) balanced biaxial tension (bulge and punch), balanced biaxial compression, and plane-strain tension. The uniaxial stress-strain behavior of 2036-T4 aluminum alloy exhibits a decreasing work-hardening rate with increasing strains which has been characterized by two-stage power law fits or by Voce-type (saturation model) equations.

Balanced biaxial experiments have taken two forms, depending on the intent. Hydraulic bulge stress-strain curves have been used for comparison with Hill's theory. Balanced biaxial prestrain by punch and die arrangements with subsequent measurement of the tensile yield stress have provided comparative work hardening curves in uniaxial tension and balanced biaxial tension. Using the first of these techniques, Pearce, Wilson, and Woodthorpe and Pearce found that commercial purity aluminum exhibits higher yield stresses in biaxial tension than predicted. The theoretical and experimental curves lie on either side of the uniaxial stress-strain curve, indicating that the yield plane is elongated in the biaxial direction with respect to the von Mises-Hill ellipse. Lloyd et al found a similar effect in 2036-T4 and 3003-O aluminum alloys, but they did not quantitatively examine...
Hill's theory. Laukonis and Ghosh\textsuperscript{19} and Sachdev\textsuperscript{21} examined balanced biaxial work hardening curves of 2036-T4 and 6009 aluminum sheet alloys, respectively, by the punch method mentioned above. In each case they found a decreased work-hardening rate in biaxial tension with respect to uniaxial tension. Using a two-stage power law fit, Laukonis and Ghosh saw a drop in \( n \) (the rate of work hardening \( d(\ln \sigma)/d(\ln \varepsilon) \)) from 0.28 to 0.22 in the strain range 0.06 \( \leq \varepsilon \leq 0.1 \) and from 0.23 to 0.17 in the range 0.15 \( \leq \varepsilon \leq 0.2 \) upon comparison of biaxial stress-strain curves with uniaxial ones. The 6009 alloy shows similar behavior with a decrease of work hardening rate under balanced biaxial tension from 0.29 to 0.23 initially (0.07 \( \leq \varepsilon \leq 0.18 \)) and from 0.24 to 0.12 at higher strains (0.18 \( \leq \varepsilon \leq 0.30 \)). However, these punch results should be applied cautiously to analysis of Hill's theory because the magnitude of the flow stress is measured in uniaxial tension after unloading from balanced biaxial tensions. Path-dependency and the Bauschinger effect may preclude accurate determination of the absolute value of the flow stress by this technique. These effects probably explain why, compared to the tensile test, lower stress-strain curves are obtained by this method while the hydraulic bulge test indicates higher stress-strain curves.

Tests of Hill's 'old' theory in plane strain deformation have generally not proved as accurate as balanced biaxial tests because of the smaller strain range attainable, the reduced sensitivity to changes in eccentricity of the yield surface, and, in some cases, the unknown frictional characteristics of the test. Taghvaipour and Mellor\textsuperscript{4} found substantial agreement with Hill's theory for stress-strain curves of commercial purity aluminum obtained in plane strain compression. Wagoner and Wang\textsuperscript{24} obtained strain distributions and loads in good agreement with Hill's theory for 2036-T4 aluminum alloy subjected to a range of strains between uniaxial tension and plane-strain tension although the strain range examined was generally below \( \varepsilon = 0.15 \).

Recently, Mellor and Parmer\textsuperscript{27-29} presented a new yield criterion (Hill's 'new' theory) for anisotropic materials which provides for another parameter, \( m \), to characterize a material's plastic behavior. The effect of \( m \) (in the 'old' Hill theory, \( m = 2 \)) is to distort the ellipse such that the new yield surface is elongated or shortened along the balanced biaxial direction while retaining the slope at \( c_2 = 0 \); i.e., preserving the proper \( r \) value. For \( m < 2.0 \), the ellipse is elongated along the \( c_2 = c_0 \) line and for \( m > 2.0 \), it is shortened along this direction. See, for example, Fig. 8(a) and 8(b) of Ref. 27.

Parmer and Mellor estimated that, for soft commercially pure aluminum sheet, \( m \) is in the range 1.7 to 1.8 based on a comparison of uniaxial and balanced biaxial work-hardening curves\textsuperscript{29} obtained by Taghvaipour and on strain distributions measured near a hole in a sheet subjected to inplane stretch.\textsuperscript{29} These results are in qualitative agreement with the above-mentioned balanced biaxial results for aluminum alloys.\textsuperscript{15,21}

It is important to note that the parameter \( m \) affects only the shape of the current yield surface. It does not allow for different work-hardening characteristics under various strain states, as has been observed in aluminum alloys.\textsuperscript{10,21} To obtain different work-hardening rates, the \( m \) value must vary with strain, violating the isotropic hardening assumption.

Plane-strain tests can provide a third independent test of Hill's 'new' theory. They can be used to establish \( m \) directly for comparison with \( m \) derived from balanced biaxial data. The approach followed here is to devise an experiment which relies on computer analysis for data interpretation but which eliminates most of the experimental uncertainties inherent in punch or hydraulic bulge tests: frictional and otherwise poorly defined boundary constraints, through-thickness strain gradients, and multistage deformation paths. The continuous evaluation of stress and strain throughout the test allows monitoring of incremental work hardening and determination of Hill's \( m \) parameter as a function of effective strain.

**EXPERIMENTAL**

Two kinds of experiments were performed: standard sheet tensile tests and sheet specimen plane-strain tests. These were carried out in a standard tensile testing machine at similar strain rates and in each case the strains were determined from photograms while loads were recorded automatically. True stress-true strain curves are directly obtainable from the tensile data but the plane-strain experiments require considerable analysis to derive the plane-strain work-hardening curves from the load and strain distribution data.

**Tensile Tests**

Tensile tests were carried out in the rolling, transverse, and 45 deg directions using ASTM E-8 tensile specimens. Each specimen was photogrided with 2.5 mm diam open circles and strained to failure at a crosshead rate of \( 8 \times 10^{-3} \) mm/s. Photographs of the grid were taken at one min intervals and the corresponding loads were recorded. Deformed circles were measured over a 7.5 mm gage length (i.e., three circles) nearest the eventual fracture location. This procedure allowed direct \( r \)-value determination as a function of strain. The average strain rate in these tests was \( 10^{-3}/s \).

**Plane-Strain Tests**

The plane-strain tests were performed in much the same way as the tensile tests described above. Eight specimen shapes (Fig. 1) were machined from the 2036-T4 aluminum sheet (1.2 mm thickness) with the rolling direction parallel to the load axis. Three types of specimens (designated B, F, H) were also machined with the transverse direction parallel to the load axis in order to check for planar anisotropy. All specimens were photogrided with either 2.5 mm open circles (three circle gage length) or 7.5 mm closed circles (one circle gage length).

These specimens fit into three categories:

1) Specimen types A through F were designed to assess the effect of varying root radius and notch angle on strain distributions. The center strains at failure are very similar among these specimens, generally between 0.09 and 0.11. The region in plane strain
(arbitrarily defined as the region along the transverse center line where \( |\epsilon_y/\epsilon_x| > 5 \) is 75 to 80 pct of the total width at the notch.

2) The \( G \)-type specimen was designed to maximize the plane-strain region while maintaining the overall dimensions between the grips. The plane-strain region is 90 pct of the width but this was achieved at the expense of fracture strains (0.08 to 0.10).

3) The \( H \)-type specimen is a result of iterative design and finite element modeling aimed at producing a large region of plane strain while maximizing the center strain at failure. The results of this effort are a specimen which fails at a center strain between 0.14 and 0.18 and exhibits plane strain (\( |\epsilon_y/\epsilon_x| > 5 \)) over approximately 80 pct of the width. The \( H \) specimens were selected for detailed comparisons of plane strain and uniaxial tensile stress-strain behavior because of the greater sensitivity allowed by the larger strain range.

The crosshead displacement rate used in these tests was \( 8 \times 10^{-4} \) mm/s, equivalent to an average axial strain rate in the plane strain region of \( 10^{-4} \) s in specimens \( A \) to \( G \) and \( 2 \times 10^{-4} \) s in the \( H \) specimens. Photographs were taken at one min intervals and loads were recorded simultaneously.

Axial and transverse grid lengths were determined along the transverse center line from photographs using a toolmaker's microscope. Details of the grid reading procedure and specimen gripping arrangement have been presented elsewhere. As noted in that earlier report, the crucial factor in the grip design is lack of slippage at the grip-specimen interface, attained after several grip design iterations.

**DATA ANALYSIS**

Three computer programs were written to analyze the experimental data obtained in the plane-strain and uniaxial tensile tests. The first of these is a general curve-fitting program for finding the required coefficients of empirical work hardening laws from stress-strain data. The second program is called the Plane-Strain Program. It interprets the grid circle and load data from the plane-strain tests taking into account the section which is near uniaxial tension, thus producing stress-strain data for the plane strain region. The third program evaluates \( m \) (Hill's 'new' parameter) as a function of strain using the curves fit to the stress-strain data obtained in plane strain and uniaxial tension. These programs are discussed briefly below. More detailed treatments appear in Appendices A thru C.

**A) General Curve Fitting Program**

The general curve-fitting program uses, as its input, digital stress-plastic strain data and a series of strain ranges to be analyzed. The following strain ranges were used for the tensile data: 0.005 to maximum, 0.01 to maximum, 0.02 to maximum, 0.01 to 0.10, 0.005 to 0.18, 0.01 to 0.18, 0.02 to 0.18, 0.05 to 0.18, 0.05 to 0.11, 0.02 to 0.07, 0.02 to 0.08, 0.02 to 0.09, 0.02 to 0.10, 0.02 to 0.11. In each of the specified strain ranges four types of curve fits are made: a generalized cubic spline (piecewise cubic function), a Hollomon\(^{25,26}\), type two-parameter power law, a Swift\(^{34,36}\)-type three-parameter power law, and a Voce\(^{37}\)-type three-parameter saturation model. In each case the optimum fit is based on minimizing the stress variance of the curve. This type of optimization avoids the excessive weighting of low strain points that occur, for example, in log-log fits. The standard deviation of each fit in each strain range is returned with the best fit coefficients.

Two additional features of this program are graphical output and calculation of incremental work hardening rates. In the graphical output, the stress-strain data are superimposed on a smooth curve representing the empirical work hardening law. This allows visual examination for scatter of the data, and for systematic deviations from the rule. The cubic spline curve is used to calculate the work hardening rate \( \eta = (d \ln \sigma)/(d \ln \epsilon) \) at 0.01 strain increments.

**B) Plane-Strain Program**

The plane-strain program uses Hill's anisotropic plasticity theory (with \( m = 2 \)) to evaluate the effective stress and strain of the plane-strain region in the plane-strain specimens. The general steps it performs are as follows:

1) For each set of grid readings (i.e., one photograph) two symmetric smooth curves are generated representing the axial and transverse strains across the specimen width. These least squares cubic splines are used for all further analysis in the program.

2) The width of the specimen along the transverse center line is divided into three regions: the center section which has a strain state close to plane strain and two edge sections near uniaxial tension. The bound-
ary of the plane strain section is set where \(|\epsilon_y/\epsilon_x| = 5\).

3) The load supported by the edge regions is calculated from Hill’s ‘old’ theory using the strain distribution data and the work hardening curve obtained from tensile tests. The error of this estimation should be small because the strain state is close to uniaxial tension and therefore deviations from Hill’s theory will be of minor importance.

4) The average effective stress in the center (plane strain) section is calculated after subtracting the estimated load supported by the edge (uniaxial) sections from the total recorded load. The remaining load is divided by the central region’s cross sectional area to obtain the average axial stress. The average effective stress in the center section is derived using Hill’s ‘old’ theory and the assumption that the center section is in plane strain.

5) The average effective strain in the center section is found by an averaging procedure based on an equal-load-supported rule described more fully in Appendix B.

It should be noted that the effective stress-effective strain curves obtained by this method are very slightly dependent on the choice of plasticity theory used to model the edge regions. This lack of sensitivity arises because of the size of these regions (~20 pct of the specimen width) and because the strain state in these regions is near uniaxial tension, where the tensile stress-strain curves are directly accurate.

The effective stress-strain curves as output from this program, do, however, use the effective stress and effective strain definitions derived from Hill’s ‘old’ theory \((m = 2)\) and may, therefore, be translated in a prescribed manner by choosing other definitions of effective stress and strain. This modification should not be confused with uncertainties involved in modelling the specimen edge region as discussed above.

### C) \(m\)-Evaluation Programs

The programs used for evaluation of Hill’s \(m\) parameter are transcendental equation solvers. Equations [C-23] and [C-25] are the ones to be solved in the rigorous analysis and simplified calculation, respectively. In the first case, the current value of \(m\) is found by solving an equation whose coefficients depend on all the previous solutions. The effective strain increment is 0.001 and \(m\) is found to an accuracy of \(10^{-5}\). In the simplified calculation, an integral is approximated by a value depending only on the total effective strain, and the same transcendental equation is solved, to within the same accuracy. In each case, the stress-strain curves in plane strain and in tension are modelled by a saturation model hardening law, in accordance with the experimental results reported here.

### RESULTS AND DISCUSSION

A full presentation of the experimental data cannot be made because of sheer numbers. These experiments resulted in more than 10,000 grid readings and the results must, therefore, appear in condensed form. The selected forms are intended to be representative and to define several important results:

1) The best form of empirical work-hardening law for 2036-T4 aluminum alloy. 2) The degree of planar anisotropy in 2036-T4 with respect to work-hardening rates and \(\bar{R}\) values. 3) The variation of work-hardening rate of 2036-T4 with strain and strain state (plane-strain or tension). 4) Application of Hill’s theory to 2036-T4, particularly with regard to yield surface shape and hardening behavior.

Figure 2 is a representative comparison of the fits between one set of tensile stress-strain data and various work-hardening laws. Of the three-parameter fits, the Voce-type equation always yielded lower standard deviations than the Swift-type equation. Little advantage was realized by introducing the third parameter in the power law equation. The Swift equation is therefore not used in comparing stress-strain curves. The cubic spline coefficients are not presented, either, because they strictly represent an arbitrarily adjustable (multicoefficient), smooth-curve fit to the data.

The reduction in work-hardening rate with increasing strain is in qualitative agreement with the two-stage power law fits employed by Laukonis and Ghosh [2036-T4] and Sachdevy [6009], and with the observations of Kocks et al.\(^{22}\) of saturation behavior of pure aluminum. There are indications,\(^{22,25}\) however, that at higher strains (above the normal tensile range) the saturation model may underestimate the work-hardening capacity of aluminum alloys.

The variation of \(\bar{R}\) with strain is presented in Fig. 3. The data points are the averages of data from three directions in the sheet (rolling direction, transverse direction, and at 45 deg to the rolling direction). The scatter within one test is as large as the scatter between tests in the three sheet directions, indicating that 2036-T4 is isotropic in the sheet plane. The uniformity of \(\bar{R}\) with strain indicates that the normal anisotropy is not a function of strain; thereby satisfying one assumption inherent in Hill’s theory.

![Graph showing comparison of fits between one set of tensile stress-strain data and various work-hardening laws.](image)

Fig. 2—Comparison of the fits, achieved by minimizing the stress variance of four types of work-hardening laws (solid lines) to one set of tensile data (points): (a) cubic spline equation, (b) saturation model (Voce), (c) three parameter power law (Swift), (d) two parameter power law (Hollomon). To separate the curves, each set of data has been displaced by approximately 33 MPa from the preceding curve. (The ordinate labels apply to curve (a)).
Figure 4 shows the axial and transverse strain distributions for an H specimen at two crosshead displacements during a plane-strain test (equivalent plots for other specimens have been presented elsewhere\textsuperscript{29}). The dots are experimental strain readings and the smooth curves represent the symmetric least-squares cubic splines used in the data analysis. The figure has been divided into three regions corresponding to the regions used in the computational procedure.

Table I is a master summary of the curve fitting done on the tensile and plane-strain data. The small strain range attainable with specimens A through G (0.02 ≤ \( \varepsilon \) ≤ 0.11) probably doesn't justify a three-parameter curve, so the best comparison of strain hardening is via the power law model. The G specimens (90 pct plane-strain region) are included in this group because no differences in stress-strain curves could be found between these and specimen types A through F. Detailed analysis of plane-strain data is based on the results obtained from the H specimens because of their greater strain capacity.

Some generalizations may be drawn from the data summarized in Table I. Within the statistical scatter, there is no difference in the work-hardening rate (\( n \)) between rolling direction and transverse direction. This is true in comparison of either plane-strain tests or tensile tests in the strain ranges permitted. There are, in fact, no indications of planar anisotropy evident from any of the tensile or plane strain results. The data from the transverse and rolling directions are, therefore, lumped to improve the statistical confidence limits.

The work hardening rate of 2036–T4 is lower in plane strain than in tension. A comparison of \( n \)-values from Table I reveals that \( n \) in plane strain is 0.222

### Table I. Summary of Work Hardening Law Coefficients from Tensile and Plane-Strain Tests

<table>
<thead>
<tr>
<th>Specimen, No.</th>
<th>Power Law Fit</th>
<th>Saturation Model Fit</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>( k )</td>
<td>( n )</td>
</tr>
<tr>
<td>Plane strain</td>
<td>B-TD (2), F-TD (1)</td>
<td>565 ± 6</td>
</tr>
<tr>
<td></td>
<td>B-RD (3), F-RD (1)</td>
<td>578 ± 14</td>
</tr>
<tr>
<td></td>
<td>All A to G (11)</td>
<td>578 ± 15</td>
</tr>
<tr>
<td>Uniaxial tension</td>
<td>RD (1), TD (1), 45 deg (1)</td>
<td>599 ± 7</td>
</tr>
<tr>
<td>Plane strain</td>
<td>H-TD (3)</td>
<td>563 ± 15</td>
</tr>
<tr>
<td></td>
<td>H-RD (2)</td>
<td>575 ± 17</td>
</tr>
<tr>
<td></td>
<td>All H (5)</td>
<td>568 ± 17</td>
</tr>
<tr>
<td>Uniaxial tension</td>
<td>RD (1), TD (1), 45 deg (1)</td>
<td>596 ± 3</td>
</tr>
</tbody>
</table>

Key:
- \( k \) Power law parameters: \( \sigma = k \varepsilon^n \).
- \( n \) Saturation model parameters: \( \sigma = S (1 - Ae^{n\varepsilon}) \).
- \( \sigma_{\text{avg}} \) Average standard deviation of curve fit to each set of stress-strain data.
- RD—Rolling direction.
- TD—Transverse direction.
- 45 deg—45 deg to rolling direction.
- ±70 pct confidence limits (i.e. one standard deviation).
to 0.224, depending on the strain range examined whereas \( n \) in tension is 0.192 to 0.209. Thus, the effect of strain state on work-hardening rate is not influenced by the strain range attainable by the specimen nor by the extent of the plane strain region.

A more detailed comparison of work-hardening rates is presented in Fig. 5, which shows the experimental variation of work-hardening rate with strain in plane-strain \((H\) specimens) and in uniaxial tension. One set of curves was generated directly from the cubic spline fit to the stress-strain data (i.e., simply smoothed data) while the other set was generated from the Voce equation coefficients presented in Table I (0.02 \( \leq r \leq 0.18 \)). The important points to notice are that the work-hardening rate is clearly lower in plane strain than in tension at every strain level, and the Voce equation well represents the variation of work hardening in both strain states.

The reduction in work-hardening rate in moving away from uniaxial tension is consistent with the effect found in balanced biaxial tests of 2036-T4 and 6009-T1 alloys. The work-hardening rate in plane strain is intermediate between the uniaxial and balanced biaxial rates.

Figure 6 is a comparison of the effective stress-effective strain behavior (represented by saturation model curves based on \( H \) specimen data and lumped tensile data) of 2036-T4 based on Hill's 'old' theory. At low strains the effective plane-strain yield stress is higher than expected, but the difference is reduced at larger strains because the work-hardening rate is lower in plane strain. At effective strains near 0.18 the plane-strain and tensile curves cross and only at that point can the yield surface shape be consistent with Hill's 'old' theory. If the theory were precise, these curves would coincide at all strains. Two divergences from the theory are indicated: 1) isotropic hardening does not hold since the work-hardening rate varies with the effective strains (Fig. 5 and 2) different effective yield stresses for different strain states at the same effective strain are found (Fig. 6).

These results imply a distortion of the yield surface from which the effective stress and effective strain definitions were derived. The second of these effects can be taken into account by introducing the constant parameter \( m \) in Hill's 'new' theory. Both effects can be represented by allowing \( m \) to vary with strain.

Figure 7 illustrates the variation of Hill's \( m \) parameter with effective strain calculated from the work hardening curves presented in Fig. 6 using both a rigorous and simplified algorithm for the computation. The analysis and computational procedure for finding \( m \) as a function of strain is presented in Appendix C. The variation of \( m \) with strain (or stress) is a transcendental function, which, when beyond the experimental strain range, shows \( m \) approaching a value of 1.49 at small strains and 2.42 at large strains (\( \geq 0.5 \)). The average value of \( m \) in the strain range tested is 1.81, in excellent agreement with previous estimates of \( m \) for commercially pure aluminum based on two other types of experiments.\(^{27,29}\)

Fig. 5—Comparison of experimental work hardening rates (points and error bars) in plane-strain and tension with equivalent curves obtained from the best-fit saturation model work hardening laws obtained in plane-strain (thin line) and uniaxial tension (thick line).

Fig. 6—Comparison of effective stress-effective strain curves obtained in uniaxial tension (thick line) and in plane-strain (thin line) using Hill's 'old' theory definitions and saturation model work hardening laws.

Fig. 7—Variation of Hill's new parameter \( m \) with strain, based on Voce-type work-hardening laws in plane-strain and uniaxial tension. (See Appendix C for calculation methods.)
Figure 8—Variation of the first quadrant of the yield surface with plastic straining based on the rigorous calculation of \( m \) with strain.

Figure 8 shows the change of yield surface (first quadrant only) with plastic straining based on the rigorous calculation of \( m \) as a function of strain depicted in Figure 7. Note that saturation occurs more rapidly in balanced biaxial tension than in uniaxial tension, indicating reduced work-hardening rates.

CONCLUSIONS

1) A new method (experimental and analytical) has been devised for measuring plane strain work hardening.

2) The method is applicable over a range of specimen geometries, eliminates several heretofore-unavoidable uncertainties, and yields results consistent with balanced biaxial results for 2036-T4 aluminum.

3) Rigorous and simplified techniques have been introduced for calculation of Hill’s \( m \) parameter as a function of strain from plane-strain and tensile stress-strain curves.

4) For 2036-T4 aluminum alloy in the strain range 0.02 to 0.18: a) The work hardening rate in plane strain is lower than in tension. b) The Hill-von Mises initial yield surface is elongated in the balanced biaxial direction. c) The above two effects can be parametrically described by a monotonic variation of Hill’s \( m \) with strain. d) A Voce-type equation represents plane strain and tensile work hardening curves well. e) There are no indications of planar anisotropy.

APPENDIX A: GENERAL CURVE FITTING

The curve fitting program fits, by the method of least squares, four types of empirical curves to a given range of true stress-true strain data:

1) Power law: \( \sigma = k\epsilon^n \) (Hollomon\(^{32,33}\))

2) Saturation model: \( \sigma = S (1 - Ae^{-B\epsilon}) \) (Voce\(^{26}\))

3) Modified power law: \( \sigma = k' (\varepsilon + \varepsilon_0)^m' \) (Swift\(^{34,35}\))

4) Cubic spline: \( \sigma = C_1\varepsilon^3 + C_2\varepsilon^2 + C_3\varepsilon + C_4 \) and, where \( X = \varepsilon - \varepsilon_t, \varepsilon_t \) is the lower limit of the particular domain to which this set of spline coefficients is fit.

In this case, a set of four cubic coefficients is generated for each strain range: 0.0 to 0.5, 0.05 to 0.10, 0.10 to 0.15, and so forth. For a typical tensile strain range of 0.0 to 0.23, a total of sixteen spline coefficients would be generated, the final strain range being 0.15 to 0.23.

The ranges of strain to be considered are user-selected and the first three curves are fit within these ranges. The cubic spline fit pertains to the whole curve. See Fig. 2 for an example of these curves.

None of the above regressions can be carried out analytically, as can polynomial least squares fits. Instead, the computer program carries out a variational procedure which minimizes the stress variance (or standard deviation) of the fit. The fit is considered optimized when the incremental change in each fit parameter is less than 10\(^{-4}\).

An incremental work hardening rate—rs—strain curve is obtained from the above-mentioned cubic spline fit. The logarithmic derivative is taken at 0.01 strain increments by calculating the stresses (from cubic spline coefficients) at strains \( \pm 0.005 \) from the strain of interest. Logarithms are taken of these stresses and strains and the slope of the line through the two points is the incremental \( n \) value.

APPENDIX B: PLANE-STRAIN DATA EVALUATION

The plane-strain data evaluation scheme presented here is based on Hill’s ‘old’ theory (for which \( m = 2 \), see Appendix C). The overall aim is to produce an effective stress-effective strain curve in plane strain for comparison with tensile data. If Hill’s ‘old’ theory were completely satisfied, these two curves would coincide; any differences being attributable to deviations from the theory. The evaluation scheme divides naturally into five steps, outlined in the text and more fully explained as follows:

1) The data from each photograph taken of the plane strain specimen consists of transverse and axial length readings at 14 to 19 positions (corresponding to the grid circle positions) along the transverse center line of the specimen and the overall axial load to which the specimen was subjected. The two sets of length readings are converted to logarithmic (true) strains based on the photograph of the unloaded specimen. A piecewise cubic function (least squares cubic spline, see Appendix A) is fit to each set of strains. This fit is forced to be symmetric by reflecting each strain through the axial center of the specimen and reproducing it in this mirror position. The overall specimen width is divided into two equal-width domains for the transverse strains and four regions for the axial strain curves. In each of these regions a cubic equation is found which best fits the strains. The boundaries between the regions are fixed and known as “knots” in the spline fit.

The grid circle data for one photograph has now been reduced to a set of four coefficients (for the half-width; the other side following from symmetry) for the transverse strains and eight coefficients for the axial strains. From these coefficients the strain state at any point on the center line can be calculated and these strains will be symmetric across the axial center line such that only half the specimen width need be analyzed.

2) Using the strain curves generated above, the point at which the absolute axial strain equals five times the
absolute transverse strain is found. This position marks the dividing line between the center (plane strain) region of the specimen and the two edge (tensile) regions.

3) The specimen half-width is swept through in twenty five equal increments. At each of these positions, the axial and transverse strains are evaluated from the cubic spline coefficients. The effective strain is calculated using Eq. [C-10a] and the expected effective stress is obtained from a three-parameter Swift type flow curve obtained from tensile tests. (A correction is made to the effective strain at each point to subtract the elastic part because the Swift-type curve was based upon purely plastic tensile strains.) The axial and transverse stress components are obtained from this effective stress using the known strain ratio via Eqs. [C-12]. This axial stress is multiplied by the current cross-sectional area of the element (= Axial/50) to obtain the axial load carried. In this manner, the total load carried by the outer sections, the load carried by the inner section, and the inner and outer areas are determined by numerical integration (i.e., by summing elemental axial loads and cross-sectional areas). The axial load expected to be carried by the inner section is used later in obtaining the average effective strain in the center section.

4) The true average axial stress in the center section is simply the measured load less the expected edge section load all divided by the current cross-sectional area of the center section. This average axial stress is converted to an average effective stress by assuming that the entire center section is sufficiently close to plane strain that Eq. [C-15a] can be applied.

5) The average effective strain in the center section is determined on an equivalent load basis. That is, the average effective strain is that effective strain which, when assumed to occur uniformly over the entire center section width, would produce the center axial load which was predicted from the measured strains using Hill's 'old' theory. This average effective strain, then, is based on Hill's 'old' theory for the averaging technique and is most simply determined from the expected axial load carried by the center section, calculated in Step 3. The average expected axial stress is simply the expected axial load divided by the current area, both calculated in Step 3. This expected axial stress is converted to expected effective stress via Eq. [C-15a], assuming plane strain in the center section. Working backward from this expected effective stress and the Swift-type flow curve, the average effective strain is obtained:

\[
\langle \varepsilon \rangle = \exp \left( \frac{\ln \bar{\sigma}_{\text{expected}} - \ln k'}{n'} - \varepsilon_0 \right).
\]  

[B-1]

Note that the average effective strain obtained in this way includes only the plastic part of the strain since the Swift fit was obtained from plastic tensile strain data.

APPENDIX C: HILL’S NEW THEORY AND ITS APPLICATION TO PLANE STRAIN

The following is a summary of the parts of Hill’s theories used in the present work and in interpreting balanced biaxial data. Hill’s ‘old’ theory results are presented as specializations of the ‘new’ theory with the m parameter fixed at 2.0. These specializations are noted by an “a” following the equation number.

Stress-Strain Relations

Hill’s new yield criterion is:

\[
2(1 + r)\tilde{\sigma}^m = (1 + 2r)(\varepsilon_1 - \varepsilon_2)^m + (\varepsilon_1 + \varepsilon_2)^m \quad \text{[C-1]}
\]

where \(\tilde{\sigma}\) is the tensile yield stress corresponding to the current stress state, and \(m\) is a constant greater than unity.

Since in these experiments the only strain states of interest lie between plane strain tension and uniaxial tension, assume that: \(\varepsilon_1 > \varepsilon_2 > 0\).

In this case, Eq. [C-1] reduces to:

\[
2(1 + r)\tilde{\sigma}^m = (1 + 2r)(\varepsilon_1 - \varepsilon_2)^m + (\varepsilon_1 + \varepsilon_2)^m \quad \text{[C-2]}
\]

By implicit differentiation, \(d\tilde{\sigma} > 0\):

\[
\frac{d\tilde{\sigma}}{d\varepsilon_1} = \frac{(1 + 2r)(\varepsilon_1 - \varepsilon_2)^{m-1} + (\varepsilon_1 + \varepsilon_2)^{m-1}}{(1 + 2r)(\varepsilon_1 - \varepsilon_2)^{m-1} - (\varepsilon_1 + \varepsilon_2)^{m-1}} \quad \text{[C-3]}
\]

but from plastic normality (with assumed positive, proportional loading):

\[
\frac{d\tilde{\sigma}}{d\varepsilon_1} = \frac{\varepsilon_2}{\varepsilon_1} \quad \text{[C-4a]}
\]

so,

\[
\varepsilon_2 = \frac{(\varepsilon_1 + \varepsilon_2)^m - 1}{(\varepsilon_1 + \varepsilon_2)^{m-1}} = \frac{(1 + 2r)(\varepsilon_1 - \varepsilon_2)^{m-1} + (\varepsilon_1 + \varepsilon_2)^{m-1}}{(1 + 2r)(\varepsilon_1 - \varepsilon_2)^{m-1} - (\varepsilon_1 + \varepsilon_2)^{m-1}} \quad \text{[C-4b]}
\]

with \(m = 2\):

\[
\varepsilon_2 = \frac{(1 + r)(\varepsilon_1 - \varepsilon_2) - \varepsilon_1}{(1 + r)(\varepsilon_1 - \varepsilon_2) + \varepsilon_1} \quad \text{[C-4d]}
\]

\[
\varepsilon_2 = \frac{(1 + r)(\varepsilon_1 - \varepsilon_2) + \varepsilon_1}{(1 + r)(\varepsilon_1 - \varepsilon_2) - \varepsilon_1} \quad \text{[C-4e]}
\]

For some common stress/strain states these relations reduce to:

a) Uniaxial tension

\[
(\sigma_2 = 0), \frac{\varepsilon_2}{\varepsilon_1} = \frac{\bar{r}}{1 + \bar{r}} \quad \text{[C-6]}
\]

for old and new Hill's theories,

b) Balanced biaxial tension

\[
(\varepsilon_1 = \varepsilon_2), \quad \varepsilon_1 = \varepsilon_2 \quad \text{[C-7]}
\]

c) Plane strain

(\varepsilon_2 = 0), \frac{\sigma_2}{\sigma_1} = \frac{k - 1}{k + 1}, \quad k = (1 + 2r)^{(1/m - 1)} \quad \text{[C-8]}

with \(m = 2\):

\[
\frac{\sigma_2}{\sigma_1} = \frac{\bar{r}}{1 + \bar{r}} \quad \text{[C-8a]}
\]

Effective Stress and Strain

To compare stress and strain states, the concepts of the equivalent (or effective) stress and equivalent strain (based on an equivalent tensile path) are intro-
duced. The equivalent stress follows directly from Eq. [C-2]:
\[ \tilde{\sigma} = \left\{ \frac{1}{2(1 + r)} \left[ (1 + 2r)(\sigma_1 - \sigma_2)^m + (\sigma_1 + \sigma_2)^m \right] \right\}^{1/m} \]  
\[ \text{[C-9]} \]

with \( m = 2 \):
\[ \tilde{\sigma} = \left[ \sigma_1^2 + \sigma_2^2 - 2(1 + r)\sigma_1\sigma_2 \right]^{1/2} \]  
\[ \text{[C-9a]} \]

with proportional, positive loading and the assumption of equivalence of plastic work (i.e., \( \delta d\tilde{\varepsilon} = \delta\sigma d\varepsilon \)):  
\[ \frac{\tilde{\sigma}}{\sigma_1} = \left( \frac{2(1 + r)}{1 + 2r} \right)^{1/m} \left[ \left( 1 - \frac{\varepsilon_2^m}{\varepsilon_1^m} \right)^{1/(m - 1)} + \left( \varepsilon_1 + \varepsilon_2 \right)^m \left( \varepsilon_1^m \right)^{1/(m - 1)} \right]^{1/(m - 1)/m} \]  
\[ \text{[C-10]} \]

\[ \tilde{\varepsilon} = \frac{1 + r}{\sqrt{1 + 2r}} \left[ \varepsilon_1^m + \varepsilon_2^m + (2r/(1 + r))\varepsilon_1 \varepsilon_2 \right]^{1/2} \]  
\[ \text{[C-10a]} \]

Introduce the factors necessary to multiply the axial stress or strain by to obtain the effective stress or strain based on Hill’s theory:
\[ f^{(m)}_{\sigma} = \frac{\tilde{\sigma}^{(m)}}{\sigma_1} \]  
\[ f^{(m)}_{\varepsilon} = \frac{\tilde{\varepsilon}^{(m)}}{\varepsilon_1} \]  
\[ \text{[C-11]} \]

where the superscript indicates the value of \( m \) in Hill’s new theory, upon which the definitions of effective stress and strain are based.

These multiplication factors can be solved generally as functions of either stress or strain ratios by combining Eqs. [C-4], [C-5], and [C-9] or [C-10]. For example, using Hill’s ‘old’ theory, if one is given \( \varepsilon_2 \) and \( \varepsilon_1 \), \( \tilde{\sigma} \) (Ref. 2) follows from Eq. [C-10a] and \( \tilde{\sigma} \) can be found from an empirical work hardening law. From this information, the axial stress can be determined:
\[ \sigma^{(2)} = \sigma_0 (1 + 2r)^{1/2} \left[ r \left( \frac{\varepsilon_2}{\varepsilon_1} \right)^2 + \frac{2r}{1 + r} \left( \frac{\varepsilon_2}{\varepsilon_1} \right) + 1 \right]^{1/2} \]  
\[ \text{[C-12]} \]

so,
\[ f^{(2)}_{\sigma} \left( \frac{\varepsilon_2}{\varepsilon_1} \right) = (1 + 2r)^{1/2} \left[ r \left( \frac{\varepsilon_2}{\varepsilon_1} \right)^2 + \frac{2r}{1 + r} \left( \frac{\varepsilon_2}{\varepsilon_1} \right) + 1 \right]^{1/2} \]  
\[ \text{[C-12]} \]

\[ \sigma_1 = \frac{\sigma^{(2)}}{f^{(2)}_{\sigma}} \].

The equivalent relations are cumbersome when using Hill’s new theory but are straightforward for some common stress and strain states:

a) Uniaxial tension (\( \varepsilon_2 = 0 \))
\[ \tilde{\sigma}^{(m)} = \sigma_1 \]  
\[ f^{(m)}_{\sigma} = 1 \]  
\[ \tilde{\varepsilon}^{(m)} = \varepsilon_1 \]  
\[ f^{(m)}_{\varepsilon} = 1 \]  
\[ \text{[C-13]} \]

by definition.

b) Balanced biaxial tension (\( \sigma_1 = \sigma_2 \))
\[ \tilde{\sigma}^{(m)} = \frac{2\sigma_1}{[2(1 + r)]^{1/m}} \]  
\[ f^{(m)}_{\sigma} = \frac{2}{[2(1 + r)]^{1/m}} \]  
\[ \tilde{\varepsilon}^{(m)} = \varepsilon_1 [2(1 + r)]^{1/m} \]  
\[ f^{(m)}_{\varepsilon} = [2(1 + r)]^{1/m} \]  
\[ \text{[C-14]} \]

or, with \( m = 2 \):
\[ f^{(2)}_{\sigma} = \sqrt{2/(1 + r)} \]  
\[ f^{(2)}_{\varepsilon} = \sqrt{2(1 + r)} \]  
\[ \text{[C-14a]} \]

These are the equations used for comparison of hydraulic bulge stress-strain curves with those generated in uniaxial tension.

c) Plane strain (\( \varepsilon_2 = 0 \))
\[ \tilde{\sigma}^{(m)} = \left[ \frac{2(m - 1)\kappa^{-1}}{(1 + \kappa)(1 + r)^{(m - 1)/(m - 1)}} \right]^{1/m} \sigma_1 \]  
\[ f^{(m)}_{\sigma} = \frac{2\kappa}{(1 + \kappa)(1 + r)^{(m - 1)/(m - 1)}} \]  
\[ \tilde{\varepsilon}^{(m)} = \left[ \frac{(1 + r)(\kappa + 1)^{(m - 1)/m}}{2(m - 1)\kappa^{-1}} \right] \varepsilon_1 \]  
\[ f^{(m)}_{\varepsilon} = \frac{(1 + r)^{(m - 1)/(m - 1)}(\kappa + 1)^{(m - 1)/m}}{2\kappa} \]  
\[ \text{[C-15]} \]

where, as before,
\[ \kappa = (1 + 2r)^{(m - 1)/(m - 1)} \]

for ‘old’ Hill theory (\( m = 2 \)):
\[ \tilde{\sigma}^{(2)} = \frac{\sqrt{1 + 2r}}{1 + r} \sigma_1, \]  
\[ f^{(2)}_{\sigma} = \frac{\sqrt{1 + 2r}}{1 + r} \]  
\[ \tilde{\varepsilon}^{(2)} = \frac{1 + r}{\sqrt{1 + 2r}} \varepsilon_1, \]  
\[ f^{(2)}_{\varepsilon} = \frac{1 + r}{\sqrt{1 + 2r}} \]  
\[ \text{[C-15a]} \]

Note that \( f^{(m)}_{\sigma} = 1/f^{(m)}_{\varepsilon} \), following naturally from the principle of equivalent plastic work.

**Relaxation of the Isotropic Hardening Assumption (Variable \( m \))**

Hill’s theories are not directly applicable to materials which exhibit differing work-hardening rates in different strain states. One way of relaxing the isotropic hardening assumption is to allow \( m \) to vary with effective strain. The necessary modifications to the Hill-theory equations are presented below and are used to obtain a rigorous plasticity theory independent of the assumption of isotropic hardening.

The only change required in allowing \( m \) to vary with strain comes about in the definition of effective strain in terms of the strain components. Variable \( m \) requires the evaluation of an integral to obtain the effective strain. Equations [C-10] and [C-11] are now correct only in differential form:
\[ d \bar{\varepsilon} = \left[ \frac{(1 + r)(k + 1)^{m - 1}}{2 - k, m - 1} \right]^{1/m} \left( \varepsilon \bar{\varepsilon}_1 \right)^{m - 1/m} \left[ (1 + 2r)^{1/(m - 1)} \right] \left\{ \varepsilon \bar{\varepsilon}_1 \right\}^{m - 1/m} \]

\[ f_\varepsilon(m) = \frac{d \bar{\varepsilon}(m)}{d \varepsilon_1} \]

for plane-strain \((d \varepsilon_2 = 0)\)

\[ d \bar{\varepsilon} = \left[ \frac{(1 + r)(k + 1)^{m - 1}}{2 - k, m - 1} \right] \left( \varepsilon \bar{\varepsilon}_1 \right)^{1/m} \frac{d \varepsilon_1}{f_\varepsilon(m)} \]

Once this plane-strain, proportional strain path has been chosen, the effective strain can be determined:

\[ \bar{\varepsilon} = \int_0^{\varepsilon_1} \left[ \frac{(1 + r)(k + 1)^{m - 1}}{2 - k, m - 1} \right]^{1/m} \left( \varepsilon \bar{\varepsilon}_1 \right)^{m - 1/m} \frac{d \varepsilon_1}{f_\varepsilon(m)} \]

This change, in allowing \(m\) to vary with \(\bar{\varepsilon}_1\), does not affect consistency or, instantaneously, Hill’s theory, because at any instant in the deformation the yield surface is defined by Hill’s equations.

Determination of \(m\) from Plane-Strain Data – 1. Rigorous Analysis

Assume that two stress-strain curves have been determined, one for uniaxial tension: \(\sigma_{\text{axial}}\) vs \(\varepsilon_{\text{axial}}\) or, equivalently, \(\sigma_{\text{tension}}\) vs \(\varepsilon_{\text{tension}}\), and one for plane strain: \(\sigma_{\text{plane}}^{(2)}\) vs \(\varepsilon_{\text{plane}}^{(2)}\), i.e., effective stress vs effective strain based on Hill’s ‘old’ theory effective stress/strain definitions, Eqs. \([C-9a]\) and \([C-10a]\). Clearly, the tensile curve is unchanged by the values of \(m\) and \(\bar{\varepsilon}_1\) since the effective stress and strain are defined to be the tensile stress and strain (Eq. \([C-13]\)). The points lying on the plane strain curve, however, can be translated by varying \(\bar{\varepsilon}_1\) and \(m\). Since \(\bar{\varepsilon}_1\) is fixed from tensile test results alone, we wish to bring the tensile and plane strain curves into coincidence by choosing the proper value of the parameter \(m\). Note that the variation of \(m\) follows from the observed variation in work-hardening rate with strain state.

Further assume that each of the two effective stress-strain curves is well represented by a Voce-type equation. That is:

\[ \sigma_{\text{tension}}^{(m)} = S_t (1 - A_t \exp[B_t \varepsilon_{\text{tension}}^{(m)}]) \]

\[ \sigma_{\text{plane}}^{(2)} = S_p (1 - A_p \exp[B_p \varepsilon_{\text{plane}}^{(2)}]) \]

where the subscript ‘\(t\)’ refers to tensile data and ‘\(p\)’ refers to plane strain data. The superscript ‘\((m)\)’ in Eq. \([C-20]\) indicates that the effective stress and strain definitions used are based on Hill’s ‘old’ theory, where \(m = 2\). The superscript ‘\((m)\)’ in Eqs. \([C-19]\) indicates that the equation is valid for any value of \(m\).

A second condition is obtained by considering Eqs. \([C-11]\), \([C-15]\), and \([C-15a]\), but with the effective strain relationships based on Eqs. \([C-17]\) and \([C-18]\). These define the relationship between effective stress and axial stress; and, between effective strain and axial strain in plane-strain tension:

\[ \sigma_{\text{axial}}^{(m)} = f_\sigma^{(m)} \varepsilon_{\text{axial}}, \quad \bar{\varepsilon}_{\text{axial}}^{(m)} = \frac{d \sigma_{\text{axial}}^{(m)}}{d \varepsilon_{\text{axial}}} \]

or, with \(m = 2\):

\[ \sigma_{\text{axial}}^{(2)} = f_\sigma^{(2)} \varepsilon_{\text{axial}}, \quad \bar{\varepsilon}_{\text{axial}}^{(2)} = \frac{d \sigma_{\text{axial}}^{(2)}}{d \varepsilon_{\text{axial}}} \]

Axial stresses and strains are the proper basis of comparison because these are the measured quantities and afford a natural invariant with respect to changes in \(m\) or \(\bar{\varepsilon}_1\). A point on the plane strain effective stress-strain curve \((m = 2)\) will map to a new point when \(m\) is varied:

\[ \left( \varepsilon_{\text{axial}}^{(2)}, \sigma_{\text{axial}}^{(2)} \right) \rightarrow \left( \varepsilon_{\text{axial}}^{(m)}, \sigma_{\text{axial}}^{(m)} \right) \]

where:

\[ \sigma_{\text{axial}}^{(m)} = \frac{f_\sigma^{(m)} f_\sigma^{(2)}}{f_\sigma^{(2)}} \bar{\varepsilon}_{\text{axial}}^{(2)} \]

\[ d \bar{\varepsilon}_{\text{axial}}^{(m)} = \frac{f_\sigma^{(m)} f_\sigma^{(2)}}{f_\sigma^{(2)}} d \bar{\varepsilon}_{\text{axial}}^{(2)} \]

\[ \bar{\varepsilon}_{\text{axial}}^{(2)} = \int_0^{\varepsilon_{\text{axial}}^{(m)}} \frac{d \varepsilon_{\text{axial}}^{(m)}}{f_\sigma^{(m)}} = \int_0^{\varepsilon_{\text{axial}}^{(m)}} \frac{d \bar{\varepsilon}_{\text{axial}}^{(m)}}{f_\sigma^{(2)}} \]

Note that, as intended, the points mapped have axial stress and axial strain as their common origin. Also note that the \(F_{\sigma}^{(m)}\) and \(F_{\sigma}^{(2)}\) are used to obtain the mapping factors between points on the curves corresponding to the two theories and that they are reciprocal.

Substituting Eqs. \([C-19]\) and \([C-20]\) into Eq. \([C-23]\) obtains:

\[ \sigma_{\text{axial}}^{(m)} = S_t \left[ 1 - A_t \exp[B_t \varepsilon_{\text{axial}}^{(m)}] \right] \]

\[ \sigma_{\text{plane}}^{(2)} = S_p \left[ 1 - A_p \exp[B_p \int_0^{\varepsilon_{\text{plane}}^{(2)}} \frac{d \varepsilon_{\text{plane}}^{(m)}}{F_{\sigma}^{(m)}} \right] \]

If the effective stress-strain curves in plane strain and tension are to be brought into coincidence, \(\sigma_{\text{plane}}^{(2)} = \sigma_{\text{plane}}^{(m)}\), \(\sigma_{\text{tension}}^{(m)}\), such that, with the substitution of \(F_{\sigma}^{(m)} = 1/F_{\sigma}^{(2)}\),

\[ S_t \left[ 1 - A_t \exp[B_t \varepsilon_{\text{tension}}^{(m)}] \right] \]

\[ = F_{\sigma}^{(m)} S_p \left[ 1 - A_p \exp[B_p \int_0^{\varepsilon_{\text{plane}}^{(m)}} \frac{d \varepsilon_{\text{plane}}^{(2)}}{F_{\sigma}^{(2)}} \right] \]

The overall governing equation to be solved is therefore:

\[ 0 = a F_{\sigma}^{(m)} \left[ 1 - A_p \beta \bar{\varepsilon}_1 \right] + \gamma \]

where:

\[ \alpha = \frac{S_p}{S_t} \]

\[ \beta = \exp[B_p] \]

\[ \gamma = A_t B_t \varepsilon_{\text{tension}}^{(m)} \]

\[ F_{\sigma}^{(m)} = \frac{1 + r}{\sqrt{1 + 2r}} \left( \frac{2k}{(1 + k)(1 + r)^{1/(m - 1)}} \right)^{(m - 1)/m} \]
\[ \chi = \frac{1}{F_e(m)} \kappa \equiv (1 + 2 \gamma)^{(1/m - 1)} \]
\[ \chi = \int_0^{\varepsilon^{(m)}} \frac{d\varepsilon^{(m)}}{F_e(m)} \approx \sum_{i=1}^{N} \frac{\varepsilon^{(m)}_i}{F_e(m)_i} \cdot N \frac{\varepsilon^{(m)}}{\delta \varepsilon^{(m)}}. \]

Note that \( \gamma \) depends on the total effective strain, \( F^{(m)}_e \), and \( \chi \) depends on the current value of \( m \) (with \( \gamma \) constant) and \( \chi \) depends on the current and past values of \( m \).

The technique for solving Eq. [C-26] requires the repeated solution of a transcendental equation which itself depends on the previous solutions. This equation may be continuously evaluated and updated to determine rigorously the relationship between \( m \) and \( \varepsilon^{(m)} \) (with the effective strain definition consistent with \( m \) at all strains). The technique used in solving Eq. [C-26] is described in the text.

**Determination of \( m \) from Plane-Strain Data – 2. Simplified Analysis**

Considerable simplification in evaluating \( m \) from tensile and plane-strain data can be made by approximating the current value of the effective strain instead of performing the integration indicated in Eqs. [C-23]. The approximation to be made is the following:

\[ \varepsilon^{(m)} \approx F^{(m)}_e \varepsilon^{(2)} \approx \frac{\varepsilon^{(2)}}{F^{(m)}_e}. \]  

This substitution would be strictly correct only for the case where isotropic hardening \( (m = \text{constant}) \) prevailed up to the strain under examination. When this substitution is made, the overall governing equation becomes:

\[ 0 = \alpha F^{(m)}_e (1 - A \beta \varepsilon^{(m)}_e) + \gamma \]

where:

\[ \alpha = \frac{S_p}{S_t} \]
\[ \beta = \exp (B_p \varepsilon^{(m)}_e) \]
\[ \gamma = A_t \exp (B_t \varepsilon^{(m)}_e) - 1 \]
\[ F^{(m)}_e = \frac{1 + \gamma}{\sqrt{1 + 2 \gamma}} \left[ \frac{2\kappa}{(1 + \kappa)(1 + \gamma)} \right]^{(m-1)/m} \]
\[ \kappa \equiv (1 + 2 \gamma)^{(1/m - 1)} \]

In this case, \( \beta \) and \( \gamma \) are simply functions of the effective strain and \( F^{(m)}_e \) represents the only \( m \) dependence. Equation [C-28] can therefore be solved numerically at any effective strain without reference to the previous solutions.

**REFERENCES**

31. C. deBoo and J. R. Rice: *"Cubic Spline Approximation I—Fixed Knots*, Computer Science Department TR20, Purdue University, April 1968.