CHAPTER 1 - PROBLEM SOLUTIONS

A. PROFICIENCY PROBLEMS

1. The plot below of load vs. extension was obtained using a specimen (shown in the following figure) of an alloy remarkably similar to the aluminum-killed steel found in automotive fenders, hoods, etc. The crosshead speed, v, was $3.3x10^{-4}$ inch/second. The extension was measured using a 2" extensioneter as shown (G). Eight points on the plastic part of the curve have been digitized for you. Use these points to help answer the following questions.



a. Determine the following quantities. Do not neglect to include proper units in your answer.

Yield stress	Young's Modulus
Ultimate tensile strength	Total elongation
Uniform elongation	Post-uniform elongation
Engineering strain rate	

b. Construct a table with the following headings, left-to-right: Extension, load, engineering strain, engineering stress, true strain, true stress. Fill in for the eight points on graph. What is the percentage difference between true and engineering strains for the first point? (i.e., $\% = ____x 100$)

What is the percentage difference between true and engineering strains for the last point?c. Plot the engineering and true stress-strain curves on a single graph using the same units.

- *d.* Calculate the work-hardening rate graphically and provide the ln-ln plot along with the value of n. How does n compare with the uniform elongation in Part a? Why?
- e. A second tensile test was carried out on an identical specimen of this material, this time using a crosshead speed of $3.3x10^{-2}$ inch/second. The load at an extension of 0.30 inch was 763.4 lb. What is the strain-rate sensitivity index, m, for this material?

SOLUTION:

a.

$$\sigma_{\rm y} = \frac{458 \text{ lbs}}{.030" \text{ x } 0.5"} = 30,500 \text{ psi } \text{E} = \frac{27,000 \text{ psi}}{0.0018"/2.0"} = 30 \text{ x } 10^6 \text{ psi}$$

$$\sigma_{\text{UTS}} = \frac{745 \text{ lbs}}{.030" \text{ x } .5"} = 49,700 \text{ psi} \qquad e_t = \frac{0.80"}{2.0"} = 0.40 \text{ or } 40\%$$
$$e_u = \frac{0.5"}{2.0"} = 0.25 \text{ or } 25\% \qquad e_{pu} = e_t - e_u = 0.40 - 0.25 = 0.15 \text{ or } 15\%$$

$$\dot{e} = \frac{3.3 \text{ x } 10^{-4 \text{ inch/s}}}{3.3"} = 10^{-4} \text{/s}$$

b.

Extension	Load	Eng. Strain	Eng. Stress	True Strain	True Stress	% Error eng/true strain
0.0018	405	0.001	27000	0.001	27024	0.04%
0.02	458	0.010	30533	0.010	30839	0.50%
0.1	630	0.050	42000	0.049	44100	2.48%
0.2	699	0.100	46600	0.095	51260	4.92%
0.3	729	0.150	48600	0.140	55890	7.33%
0.4	741.5	0.200	49433	0.182	59320	9.70%
0.5	745	0.250	49667	0.223	62083	12.04%
0.8	440	0.400	29333	0.336	41067	18.88%

c.



d.



The n value (slope of the ln-ln plot) is as follows: All points: n = 0.13, Wth first and last removed: n = 0.225

The first point must be removed because the elastic strain is a significant part of the total strain and the last point is meaningless because necking means that the formula to find ε_t , and σ_t cannot be used.

.225 differs from 0.25 because n is true strain so $e^{.225} - 1 = 0.25 = uniform$ elongation.

$$m = \frac{\ln p_2/p_1}{\ln v_2/v_1} = \frac{\ln \frac{763.4 \text{ lb}}{729 \text{ lb}}}{\ln \frac{3.3 \times 10^{-2}/\text{s}}{3.3 \times 10^{-4}/\text{s}}} = \frac{\ln 1.047}{\ln 100} = \frac{.046}{4.605} = 0.010$$

e.

- 2. Starting from the basic idea that tensile necking begins at the maximum load point, find the true strain and engineering strain where necking begins for the following material laws. Derive a general expression for the form and find the actual strains.
 - a. $\sigma = k (\varepsilon + \varepsilon_o)^n$ $\sigma = 500 (\varepsilon + 0.05)^{0.25}$ (Swift)¹b. $\sigma = \sigma_o + k (\varepsilon + \varepsilon_o)^n$ $\sigma = 100 + 500 (\varepsilon + 0.05)^{0.25}$ (Ludwik)c. $\sigma = \sigma_o (1 A_e^{-B\varepsilon})$ $\sigma = 500 [1 0.6 \exp(-3\varepsilon)]$ (Voce)²d. $\sigma = \sigma_o$ $\sigma = 500 [1 0.6 \exp(-3\varepsilon)]$ (Ideal)e. $\sigma = \sigma_o + k\varepsilon$ $\sigma = 250 + 350 \varepsilon$ (Linear)f. $\sigma = k \sin (B\varepsilon)$ $\sigma = 500 \sin (2\pi\varepsilon)$ (Trig)

SOLUTION:

a. $\sigma = k (\varepsilon + \varepsilon_0)^n$

$$\frac{d\sigma}{d\epsilon} = nk (\epsilon + \epsilon_o)^{n-1} = k(\epsilon + \epsilon_o)^n = \sigma$$

$$n = \epsilon_u + \epsilon_o, \quad \boxed{\epsilon_u = n - \epsilon_o}$$
for $\epsilon_o = 0.05, \quad n = 0.25 \quad \boxed{\epsilon_u = 0.20}$

b.

$$\frac{d\sigma}{d\epsilon} = nk \left(\epsilon + \epsilon_{o}\right)^{n-1} = \sigma_{o} + k(\epsilon + \epsilon_{o})^{n} = \sigma$$

 $\sigma_{o} + k(\varepsilon + \varepsilon_{o})^{n-1} [\varepsilon + \varepsilon_{o} - n] = 0$

 $\sigma = \sigma_{o} + k(\varepsilon + \varepsilon_{o})^{n}$

This is transcendental, so it cannot be solved algebraically.

Let's solve it numerically by Newton's Method for the special case n = 0.25, $\varepsilon_o = 0.05$, $\sigma_o = 100$, k = 500.

¹ H. W. Swift: *Plastic Instability Under Plane Stress*, <u>J. Mech. Phys. Solids</u>, 1952, Vol. 1, p.1.

 ² E. Voce: The Relationship Between Stress and Strain For Homogeneous Deformation, J. Inst. Met: 1948, Vol. 74, p. 537-562, 760.
 E. Voce: <u>The Engineer</u>: 1953, Vol. 195, p.23.

E. Voce: Metallurga: 1953, Vol. 51, p. 219.

$$F(\varepsilon) = \sigma_{o} + k(\varepsilon + \varepsilon_{o})^{n-1} \left[\varepsilon + \varepsilon_{o} - n\right] = 0$$

$$\mathbf{F}'(\boldsymbol{\varepsilon}) = \mathbf{k}(\mathbf{n}-2)(\boldsymbol{\varepsilon} + \boldsymbol{\varepsilon}_{\mathrm{o}})^{\mathbf{n}-2} \left[\boldsymbol{\varepsilon} + \boldsymbol{\varepsilon}_{\mathrm{o}} - \mathbf{n}\right] + \mathbf{k}(\boldsymbol{\varepsilon} + \boldsymbol{\varepsilon}_{\mathrm{o}})^{\mathbf{n}-1}$$

Start from a trial of $e_u = 0.20$ (from Part b)

Step (i)	$\epsilon_u(i)$	$F[\epsilon_u(i)]$	$F'[\varepsilon_u(i)]$	$\epsilon_u(i+1)$
0	0.20	100	1,414	0.129
1	0.129	-29	3,078	0.138
2	0.138	-8.5	2,762	0.142

So,
$$e_u \approx 0.142$$

c. $\sigma = \sigma_0 (1 - Ae^{-B\epsilon})$

 $\frac{d\sigma}{d\epsilon} = BA\sigma_{o}e^{-B\epsilon} = \sigma_{o}(1 - Ae^{-B\epsilon}) = \sigma$

BX = 1-X where $X = Ae^{-B\epsilon}$

$$X = \frac{1}{1+B} \text{ or } \ln X = \ln \frac{1}{1+B}, \ln A - B\epsilon = \ln \frac{1}{1+B}, -B\epsilon = \ln \frac{1}{1+B},$$
$$-B\epsilon = \ln \frac{1}{1+B} - \ln A$$
$$\boxed{\epsilon_{u} = -\frac{1}{B} \left[\ln A - \ln \frac{1}{1+B} \right] = \frac{1}{B} \ln A(1+B)}$$

for A = 0.6 B = 3: $\epsilon_u = \frac{1}{3} \ln [0.6(4)] = 0.29$

d.
$$\sigma = \sigma_{o_1} \frac{d\sigma}{d\epsilon} = 0 = \sigma_o = \sigma$$
 (Never stable for constant σ_{o_1} . $\overline{\epsilon_u = 0}$)

e.
$$\sigma = \sigma_0 + k\epsilon$$
, $\frac{d\sigma}{d\epsilon} = k = \sigma_0 + k\epsilon = \sigma$, $\epsilon = \frac{k - \sigma_0}{k}$

for
$$\sigma_0 = 250$$
, k = 350, $\varepsilon_u = \frac{350 - 250}{350} = 0.29$

f. $\sigma = k \sin (B\epsilon)$

$$\frac{d\sigma}{d\varepsilon} = kB \cos \left(B\varepsilon\right) = k \sin \left(B\varepsilon\right), \qquad B = \tan B\varepsilon, \qquad \varepsilon = \frac{1}{B} \tan^{-1} E$$

for B =
$$2\pi$$
, k = 500, $\varepsilon = \frac{1}{2\pi} \tan^{-1} 2\pi = 0.22$

3. What effect does a multiplicative strength coefficient (for example k in the Hollomon Law, k in Problem 2.a., or σ_0 in Problem 2.c.) have on the uniform elongation?

SOLUTION:

No effect. Because it is only the ratio of strength in one part of the tensile test (i.e. in the neck) to another (outside the neck), multiplication of σ has no effect on stability.

- For each of the explicit hardening laws presented in Problem 2, calculate the true stress at $\varepsilon =$ 4. 0.05, 0.10, 0.15, 0.20, 0.25 and plot the results on a (ln σ -ln ε) figure. Use the figure to calculate a best-fit n value for each material and compare this with the uniform strain calculated in Problem 2. Why are they different, in view of Eq. 1.16?
- 5. For each of the explicit hardening laws presented in Problem 2, plot the engineering stress-strain curves and determine the maximum load point graphically. How do the results from this procedure compare with those obtained in Problems 2 and 4?

SOLUTIONS:

See table and plots. Compare ε_u and n from ln-ln plots

Equation	ε _u (Problem 2)	ε _u (Problem 4)	ε _u (Problem 5) (from max load)
a	0.20	0.17	0.20
b	0.14	0.13	0.14
с	0.29	0.24	0.29
d	0.00	0.00	0.00
e	0.29	0.14	0.29
f	0.22	0.75	0.22

The results are different from Problems 2 and 4 because $\frac{d \ln \sigma}{d \ln \epsilon}$ (=n) is not a constant. Only this quantity dσ $= \sigma$

is important, not an average of this quantity over a large range of strains. at the point at which $d\epsilon$

The results from Problems 2 and 5 are identical, whether Considere's Criterion is used mathematically (Problem 2) or whether the hardening equation is plotted in engineering units and the maximum load is found.

$$\frac{\sigma_2}{\sigma_1} \text{ (at two rates)} = \frac{520 (\epsilon + 0.05)^{0.25}}{500 (\epsilon + 0.05)^{0.25}} = \frac{520}{500}$$

			Proble	em 1-4		
	Stress	Stress	Stress	Stress	Stress	Stress
Strain	Part a	Part b	Part c	Part d	Part e	Part f
0.05	281	381	242	500	268	155
0.1	311	411	278	500	285	294
0.15	334	434	309	500	303	405
0.2	354	454	335	500	320	476
0.25	370	470	358	500	338	500
	In stress	ln stress	ln stress	ln stress	In stress	ln stress
ln strain	Part a	Part b	Part c	Part d	Part e	Part f
-2.996	5.639	5.943	5.488	6.215	5.589	5.040
-2.303	5.740	6.019	5.627	6.215	5.652	5.683
-1.897	5.812	6.074	5.732	6.215	5.712	6.003
-1.609	5.868	6.117	5.815	6.215	5.768	6.164
-1.386	5.914	6.153	5.881	6.215	5.822	6.215
slope (n)	0.17	0.13	0.24	0.00	0.14	0.75

(Figure for Problem 1-4 follows.)



ln (true strain)



Figure for Problem 1-4 (upper), for Problem 1-5 (lower).

Problem 1-5							
True	Eng.	Eng. Stress					
Strain	Strain	Part a	Part b	Part c	Part d	Part e	Part f
0.01	0.01	245.0	344.0	206.8	495.0	251.0	31.1
0.02	0.02	252.1	350.1	213.2	490.1	251.9	61.4
0.03	0.03	258.1	355.1	219.1	485.2	252.8	90.9
0.04	0.04	263.1	359.2	224.8	480.4	253.6	119.5
0.05	0.05	267.5	362.6	230.0	475.6	254.5	147.0
0.06	0.06	271.2	365.4	234.9	470.9	255.2	173.3
0.07	0.07	274.4	367.6	239.5	466.2	255.9	198.5
0.08	0.08	277.1	369.5	243.7	461.6	256.6	222.4
0.09	0.09	279.5	370.9	247.7	457.0	257.3	244.9
0.1	0.11	281.6	372.0	251.3	452.4	257.9	265.9
0.11	0.12	283.3	372.9	254.7	447.9	258.4	285.5
0.12	0.13	284.8	373.4	257.8	443.5	259.0	303.6
0.13	0.14	286.0	373.8	260.7	439.0	259.5	320.1
0.14	0.15	287.0	373.9	263.3	434.7	259.9	334.9
0.15	0.16	287.8	373.9	265.7	430.4	260.4	348.2
0.16	0.17	288.4	373.6	267.9	426.1	260.8	359.7
0.17	0.19	288.9	373.3	269.8	421.8	261.1	369.7
0.18	0.20	289.2	372.7	271.6	417.6	261.4	377.9
0.19	0.21	289.4	372.1	273.2	413.5	261.7	384.4
0.2	0.22	289.5	371.3	274.6	409.4	262.0	389.3

0.21	0.23	289.4	370.5	275.8	405.3	262.2	392.6
0.22	0.25	289.2	369.5	276.8	401.3	262.4	394.2
0.23	0.26	289.0	368.4	277.7	397.3	262.6	394.1
0.24	0.27	288.6	367.3	278.4	393.3	262.7	392.5
0.25	0.28	288.2	366.1	279.0	389.4	262.8	389.4
0.26	0.30	287.7	364.8	279.5	385.5	262.9	384.8
0.27	0.31	287.1	363.4	279.8	381.7	263.0	378.7
0.28	0.32	286.4	362.0	280.0	377.9	263.0	371.2
0.29	0.34	285.7	360.5	280.1	374.1	263.0	362.4
0.3	0.35	284.9	359.0	280.1	370.4	263.0	352.3
0.31	0.36	284.1	357.4	279.9	366.7	262.9	341.0
0.32	0.38	283.2	355.8	279.7	363.1	262.9	328.5
0.33	0.39	282.2	354.1	279.3	359.5	262.8	315.0
0.34	0.40	281.2	352.4	278.9	355.9	262.6	300.5
0.35	0.42	280.2	350.7	278.4	352.3	262.5	285.1
0.36	0.43	279.1	348.9	277.8	348.8	262.3	268.8
Uniform st	rain (eng.)	0.22	0.15	0.34	0.00	0.33	0.25
Uniform St	rain (true)	0.20	0.14	0.29	0.00	0.29	0.22

6. Tensile tests at two crosshead speeds (1mm/sec and 10mm/sec) can be fit to the following hardening laws:

at
$$V_1 = 1$$
mm/sec, $\sigma = 500 (\varepsilon + 0.05)^{0.25}$
at $V_2 = 10$ mm/sec, $\sigma = 520 (\varepsilon + 0.05)^{0.25}$

What is the strain-rate sensitivity index for these two materials? Does it vary with strain? What is the uniform strain of each, according to the Considere Criterion?

SOLUTION:

$$m = \frac{\ln (\sigma_2 / \sigma_1)}{\ln (v_2 / v_1)} = \frac{\ln (520 / 500)}{\ln (10 / 1)} = 0.017$$

The strain-rate sensitivity is independent of strain because the ratio of stresses at the two strain rates is independent of strain.

Substituting into the result for Problem 2a gives the uniform true strain in each case:

$$\varepsilon_{n}(v_{2}) = \varepsilon_{n}(v_{1}) = n - \varepsilon_{0} = .25 - 0.05 = 0.20$$

7. Repeat Problem 6 with two other stress-strain curves:

at
$$V_1 = 1$$
 mm/sec, $\sigma = 550 \epsilon^{0.25}$

at V₂ = 10mm/sec,
$$\sigma = 500 \epsilon^{0.20}$$

Plot the stress-strain curves and find the strain-rate sensitivity index at strains of 0.05, 0.15, and 0.25. In view of these results, does Eq. 1.17 apply to this material?

SOLUTION:

$$\frac{\sigma_2}{\sigma_1} = \frac{550 \,\epsilon^{0.25}}{500 \,\epsilon^{0.20}} = 1.1 \,\epsilon^{0.05} \qquad m = \frac{\ln 1.1 \,\epsilon^{0.05}}{\ln 10} = \frac{0.095 + 0.05 \ln \epsilon}{2.303} = 0.041 + 0.022 \ln \epsilon$$



In this case, the strain-rate sensitivity varies with strain.

Eq. 1.17 applies equally well to Problem 6 or Problem 7 at a given strain rate. The difficulty is that the equation was derived assuming that tensile stress depends only on tensile strain. However, the effect of strain-rate sensitivity on the maximum load point is small if m << n, as is the usual case. However, the *post-uniform elongation* depends strongly on even small values of m.

8. Consider the engineering stress-strain curves for three materials labeled A, B, and C below. Qualitatively, put the materials in order in terms of largest-to-smallest strain hardening (n-value) and strain-rate sensitivity (m-value).



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SOLUTION: Strain hardening (based on strain to maximum load) order: B, A, C.

Strain-rate sensitivity (based on post-uniform strain) order: A, C, B.

Ductility or formability (based on total strain to failure) order: A, B, C.

9. It is very difficult to match tensile specimens precisely. For sheet materials, the thickness, width, and strength may vary to cause a combined uncertainty of about $\pm 1\%$ in stress. Considering this uncertainty of K's in Problem 6, calculate the range



of m values which one might obtain if one conducted the tests at both rates several times.

SOLUTION:

From Problem 6, we recall that

 $m = \frac{\ln \frac{520}{500}}{\ln 10} = 0.017$, but now we consider the range: $520 \pm 1\% \times 520 = 515$ to 525and $500 \pm 1\% \times 500 = 495$ to 505 $\begin{cases} m_{low} = \frac{\ln \frac{515}{505}}{\ln 10} = 0.009\\ m_{high} = \frac{\ln \frac{525}{495}}{\ln 10} = 0.026 \end{cases} = 0.017 \pm 0.009$

So, the combined uncertainty of m is in the range:

So, a $\pm 1\%$ uncertainty in stress corresponds to a $\pm 50\%$ uncertainty in m!

10. Considering the specimen-to-specimen variation mentioned in Problem 9, it would be very desirable to test strain-rate sensitivity using a single specimen. Typically, "jump-rate tests" are conducted by abruptly changing the crosshead velocity during the test. Find the strain-rate sensitivity for the idealized result shown:

SOLUTION:

$$m = \frac{\ln \frac{315}{300}}{\ln 10} = 0.021$$

11. In fact, the procedure outlined in Problem 10, while being convenient and attractive, has its own difficulties. In order to obtain sufficient resolution of stress, it is necessary to expand the range and to move the zero point. Some equipment does not have this capability. More importantly, the response shown in Problem 10 is not usual. For the two more realistic jump-rate tests reproduced

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below, find m values using the various points marked.



SOLUTION:

For the "up jump" in rate:

$$m_{\rm B} = \frac{\ln \frac{315}{300}}{\ln 10} = 0.021 \qquad \qquad m_{\rm C} = \frac{\ln \frac{330}{300}}{\ln 10} = 0.041 \qquad \qquad m_{\rm D} = \frac{\ln \frac{345}{300}}{\ln 10} = 0.061$$

For the "down jump" in rate:

$$m_{\rm B} = \frac{\ln \frac{310}{315}}{\ln \frac{1}{10}} = 0.007 \qquad \qquad m_{\rm C} = \frac{\ln \frac{300}{315}}{\ln \frac{1}{10}} = 0.021 \qquad \qquad m_{\rm D} = \frac{\ln \frac{290}{315}}{\ln \frac{1}{10}} = 0.036$$

It should be apparent that neither the jump or continuous method eliminates the uncertainties.

B. DEPTH PROBLEMS

12. If a tensile test specimen were not exactly uniform in cross section, for example if there were

initial tapers as shown below, how would you expect the measured true stress-strain curves to appear relative to one generated from a uniform specimen? Sketch the stress-strain curves you would expect.



SOLUTION:

The presence of a notch tends to concentrate the strain in the reduced gage section such that work hardening occurs there rapidly. In a more severe notch, the stress state begins to have a lateral component (tending toward plane strain) which leads to more hardening. Therefore, one might expect the behavior to appear as shown.



13. What is the relevance of the 0.2% offset in determining the engineering yield stress?

SOLUTION:

It is simply a convenient number; small enough so that little strain hardening takes place but large

enough to resolve using most tensile testing equipment.

14. Some low-cost steels exhibit tensile stress-strain curves as shown below. What would you expect to happen with regard to necking?



SOLUTION:

During the first, flat stage one should expect localization to begin. In fact, this happens in a narrow band called a Luder's band, but as the strain there increases the material in the bank increases and the flow stress exceeds that of the surrounding material. The bank thus moves outward until the entire specimen is strained beyond the flat region. After that, straining takes place normally.

15. It has been proposed that some materials follow a tensile constitutive equation which has additive effects of strain hardening and strain-rate hardening rather than multiplicative ones:

> multiplicative: $\sigma = F(\varepsilon) G(\varepsilon)$ additive: $\sigma = F(\varepsilon) + G(\dot{\varepsilon})$

In the first case one investigates $G(\dot{\epsilon})$ at constant ϵ by examining $\frac{\sigma(V_2)}{\sigma(V_1)}$, as we have done so far. In the second case, one would watch $\sigma(V_2) - \sigma(V_1)$. Assume that an additive law of the following *type were followed by a material:*

$$\sigma = 500 \ \epsilon^{0.25} \ + 25 \left(\frac{\dot{\epsilon}}{\dot{\epsilon}_o}\right)^{0.030}$$

where $\hat{\mathbf{e}}_{0}$ is the base strain-rate where the strain hardening law is determined (i.e. a tensile test conducted at a strain rate of $\dot{\varepsilon}_0$ exhibits $\sigma = 500 \varepsilon^{0.25}$).

- a. Given this law, determine the usual multiplicative m value at various strains from two tensile tests, one conducted at $\dot{\varepsilon}_0$ and one at $10\dot{\varepsilon}_0$.
- b. Compare tensile results extracted from the additive law provided and the multiplicative one determined in Part a. [Use the m value obtained from the center of the strain range, at $\varepsilon = 0.125.1$

SOLUTION:

$$\sigma = 500 \ \varepsilon^{0.25} + 25 \left(\frac{\varepsilon}{\varepsilon_{0}}\right)^{0.30}$$

at $\varepsilon = \varepsilon_{0}$ $\sigma = 500 \ \varepsilon^{0.25} + 25$
at $\varepsilon = 10 \ \varepsilon_{0}$ $\sigma = 500 \ \varepsilon^{0.25} + 25 \ (10)^{0.30} = 500 \ \varepsilon^{0.25} + 50$
$$m = \frac{\ln \frac{500 \ \varepsilon^{0.25} + 50}{500 \ \varepsilon^{0.25} + 25}}{\ln \ 10}, \text{ such that at} \begin{cases} \varepsilon = 0.05 \ m = 0.040 \\ \varepsilon = 0.15 \ m = 0.031 \\ \varepsilon = 0.125 \ m = 0.032 \end{cases}$$

The m value decreases with strain because the stress difference between the two rates is reduced relative to the overall flow stress.

16. Use Eq. 1.1-19 (or, equivalently, Eqs. 1.1-20 and 1.1-22) to find the plastic instability for the strain hardening $[f(\varepsilon)]$ and strain-rate hardening $[g(\varepsilon)]$ forms specified. In each case m=0.02 and $\dot{\varepsilon}_0=1/sec$.

a.
$$\sigma = f(\varepsilon) g(\dot{\varepsilon}), f(\varepsilon) from Problem 2a, g(\dot{\varepsilon}) = \left(\frac{\dot{\varepsilon}}{\dot{\varepsilon}_{o}}\right)^{m}$$

b. $\sigma = f(\varepsilon) g(\dot{\varepsilon}), f(\varepsilon) from Problem 2c, g(\dot{\varepsilon}) = \left(\frac{\dot{\varepsilon}}{\dot{\varepsilon}_{o}}\right)^{m}$
c. $\sigma = f(\varepsilon) g(\dot{\varepsilon}), f(\varepsilon) from Problem 2d, g(\dot{\varepsilon}) = \left(\frac{\dot{\varepsilon}}{\dot{\varepsilon}_{o}}\right)^{m}$
d. $\sigma = f(\varepsilon) g(\dot{\varepsilon}), f(\varepsilon) from Problem 2e, g(\dot{\varepsilon}) = \left(\frac{\dot{\varepsilon}}{\dot{\varepsilon}_{o}}\right)^{m}$
e. $\sigma = f(\varepsilon) g(\dot{\varepsilon}), f(\varepsilon) from Problem 2f, g(\dot{\varepsilon}) = \left(\frac{\dot{\varepsilon}}{\dot{\varepsilon}_{o}}\right)^{m}$
f. $\sigma = f(\varepsilon) + g(\dot{\varepsilon}), f(\varepsilon) and g(\dot{\varepsilon}) from Problem 15. (Leave Part f in equation form.)$

SOLUTION:

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a.

$$\sigma = k \left(\varepsilon + \varepsilon_{o}\right)^{n} \left(\frac{\dot{\varepsilon}}{\dot{\varepsilon}_{0}}\right)^{m}$$

$$n_{i} = \left(\frac{\partial \ln \sigma}{\partial \ln \varepsilon}\right)_{\dot{\varepsilon}}^{i} = \frac{n\varepsilon}{\varepsilon + \varepsilon_{o}}, \quad m_{i} = \left(\frac{\partial \ln \sigma}{\partial \ln \dot{\varepsilon}}\right)_{\varepsilon}^{i} = m$$

$$\varepsilon = \frac{n\varepsilon}{(1-m)(\varepsilon + \varepsilon_{o})} \implies \varepsilon_{u} = \frac{n}{1-m} - \varepsilon_{o} = 0.205$$

b.

c.

d.

$$\sigma = \sigma_{o} \left(1 - A_{e}^{-Be}\right) \left(\frac{\dot{e}}{\dot{e}_{0}}\right)^{m}$$

$$n_{i} = \left(\frac{\partial \ln \sigma}{\partial \ln \epsilon}\right)_{\dot{\epsilon}} = \frac{BA\epsilon \exp(-B\epsilon)}{1 - A \exp(-B\epsilon)}, \quad m_{i} = \left(\frac{\partial \ln \sigma}{\partial \ln \epsilon}\right)_{\epsilon} = m$$

$$\overline{\epsilon_{u}} = \frac{1}{B} \ln \left[\frac{A (B+1-m)}{1-m}\right] = 0.300$$

$$\sigma = \sigma_{o} \left(\frac{\dot{e}}{\dot{\epsilon}_{0}}\right)^{m}$$

$$n_{i} = \left(\frac{\partial \ln \sigma}{\partial \ln \epsilon}\right)_{\dot{\epsilon}} = 0, \quad m_{i} = \left(\frac{\partial \ln \sigma}{\partial \ln \epsilon}\right)_{e} = m$$

$$\overline{\epsilon_{u}} = 0 \quad (\text{Never stable})$$

$$\sigma = \left(\sigma_{o} + k\epsilon\right) \left(\frac{\dot{e}}{\dot{\epsilon}_{0}}\right)^{m}$$

$$n_{i} = \left(\frac{\partial \ln \sigma}{\partial \ln \epsilon}\right)_{\dot{\epsilon}} = \frac{k\epsilon}{\sigma_{o} + k\epsilon}, \quad m_{i} = \left(\frac{\partial \ln \sigma}{\partial \ln \epsilon}\right)_{\epsilon} = m$$

$$\overline{\epsilon} = \frac{k\epsilon}{(1-m)(\sigma_{o} + k\epsilon)} \implies \overline{\epsilon_{u} = \frac{1}{1-m} - \frac{\sigma_{o}}{k} = 0.306}$$

$$\sigma = k \sin (B\epsilon) \left(\frac{\dot{\epsilon}}{\dot{\epsilon}_{0}}\right)^{m}$$

$$n_{i} = \left(\frac{\partial \ln \sigma}{\partial \ln \epsilon}\right)_{\dot{\epsilon}} = \frac{B\epsilon}{\tan(B\epsilon)}, \quad m_{i} = \left(\frac{\partial \ln \sigma}{\partial \ln \dot{\epsilon}}\right)_{\epsilon} = m$$

$$\overline{\epsilon} = \frac{B\epsilon}{(1-m)\tan(B\epsilon)} \implies \overline{\epsilon_{u}} = \frac{1}{B} \tan^{-1}\left(\frac{B}{1-m}\right) = 0.225$$

$$\begin{split} \sigma &= k \; \epsilon^n + B \!\! \left(\! \frac{\dot{\epsilon}}{\dot{\epsilon}_o} \! \right)^m \\ n_i &= \! \left(\! \frac{\partial ln\sigma}{\partial ln\epsilon} \! \right)_{\!\! \hat{\epsilon}} = \! \frac{nk\epsilon^n}{k\epsilon^n + B \! \left(\! \frac{\dot{\epsilon}}{\dot{\epsilon}_o} \! \right)^m} \,, \quad m_i = \! \left(\! \frac{\partial ln\sigma}{\partial ln\epsilon} \! \right)_{\!\! \epsilon} = \! \frac{Bm \! \left(\! \frac{\dot{\epsilon}}{\dot{\epsilon}_o} \! \right)^m}{k\epsilon^n + B \! \left(\! \frac{\dot{\epsilon}}{\dot{\epsilon}_o} \! \right)^m} \end{split}$$

Substitution leads to a transcendental equation:

$$n\varepsilon^{n-1} - \varepsilon^n = \frac{B}{k} (1-m) \left(\frac{\dot{\varepsilon}}{\dot{\varepsilon}_0}\right)^m$$
,

which may be solved iteratively if so desired. Note that for an additive law such as this one, the plastic instability strain depends on strain rate as well as material constants.

CHAPTER 2 - PROBLEM SOLUTIONS

A. PROFICIENCY PROBLEMS

1. Perform the indicated vector operations using the vector components provided:

a↔	\rightarrow (1, 1, 1) b \leftrightarrow (1, 2, 3)	$\mathbf{c} \leftrightarrow (-1, 1, -1)$
a∙b	a×b	a · (b×c)
a∙c	a×c	$(\mathbf{a} \times \mathbf{b}) \cdot (\mathbf{a} \times \mathbf{c})$
b∙c	b×c	$\mathbf{a} \cdot (\mathbf{b} + \mathbf{c})$
a+b	b×a	$\mathbf{a} \cdot \mathbf{b} + \mathbf{a} \cdot \mathbf{c}$
a+c	c×a	a ×(b + c)
b+c	c×b	$(\mathbf{a} \times \mathbf{b}) + (\mathbf{a} \times \mathbf{c})$

SOLUTION:

Note:

$$\mathbf{a} \cdot \mathbf{b} \leftrightarrow \mathbf{a}_{1} \mathbf{b}_{1} = \mathbf{a}_{1}\mathbf{b}_{1} + \mathbf{a}_{2}\mathbf{b}_{2} + \mathbf{a}_{3}\mathbf{b}_{3}$$

$$\mathbf{a} \times \mathbf{b} = \varepsilon_{kij} \mathbf{a}_{1} \mathbf{b}_{j} \mathbf{x}_{k} = (\mathbf{a}_{2}\mathbf{b}_{3} - \mathbf{a}_{3}\mathbf{b}_{2}) \mathbf{x}_{1} + (\mathbf{a}_{3}\mathbf{b}_{1} - \mathbf{a}_{1}\mathbf{b}_{3}) \mathbf{x}_{2} + (\mathbf{a}_{1}\mathbf{b}_{2} - \mathbf{a}_{2}\mathbf{b}_{1}) \mathbf{x}_{3}$$

$$\mathbf{a} \cdot \mathbf{b} = (1, 1, 1) \cdot (1, 2, 3) = 1 \cdot 1 + 1 \cdot 2 + 1 \cdot 3 = 6$$

$$\mathbf{a} \cdot \mathbf{c} = (1, 1, 1) \cdot (-1, 1, -1) = -1 + 1 - 1 = -1$$

$$\mathbf{b} \cdot \mathbf{c} = (1, 2, 3) \cdot (-1, 1, -1) = -1 + 2 - 3 = -2$$

$$\mathbf{a} + \mathbf{b} = (1, 1, 1) + (1, 2, 3) \leftrightarrow (2, 3, 4)$$

$$\mathbf{a} + \mathbf{c} = (1, 1, 1) + (-1, 1, -1) \leftrightarrow (0, 2, 0)$$

$$\mathbf{b} + \mathbf{c} = (1, 2, 3) + (-1, 1, -1) \leftrightarrow (0, 3, 2)$$

$$\mathbf{a} \times \mathbf{b} = \begin{vmatrix} \mathbf{\hat{x}}_{1} \cdot \mathbf{\hat{x}}_{2} \cdot \mathbf{\hat{x}}_{3} \\ 1 \cdot 1 \cdot 1 \\ 1 \cdot 2 \cdot 3 \end{vmatrix} = \begin{vmatrix} 1 \cdot 1 \\ 2 \cdot 3 \end{vmatrix} \begin{vmatrix} \mathbf{\hat{x}}_{1} - \begin{vmatrix} 1 \cdot 1 \\ 1 \cdot 3 \end{vmatrix} \begin{vmatrix} \mathbf{\hat{x}}_{2} + \begin{vmatrix} 1 \cdot 1 \\ 1 \cdot 2 \end{vmatrix} \begin{vmatrix} \mathbf{\hat{x}}_{3} - \mathbf{\hat{x}}_{1} - \mathbf{\hat{x}}_{2} + \mathbf{\hat{x}}_{3} \leftrightarrow (1, -2, 1) \end{vmatrix}$$

In a similar way,

 $\mathbf{a} \times \mathbf{c} \leftrightarrow (1, -2, 1)$ $\mathbf{b} \times \mathbf{c} \leftrightarrow (-2, 0, 2)$ $\mathbf{b} \times \mathbf{a} \leftrightarrow (-5, -2, 3)$ $\mathbf{c} \times \mathbf{a} \leftrightarrow (2, 0, -2)$ $\mathbf{c} \times \mathbf{b} \leftrightarrow (5, 2, -3)$

 $\mathbf{a} \cdot (\mathbf{b} \times \mathbf{c}) = -4$ $(\mathbf{a} \times \mathbf{b}) \cdot (\mathbf{a} \times \mathbf{c}) = 0$ $\mathbf{a} \cdot (\mathbf{b} + \mathbf{c}) = 5$ $\mathbf{a} \cdot \mathbf{b} + \mathbf{a} \cdot \mathbf{c} = 5$ $\mathbf{a} \times (\mathbf{b} + \mathbf{c}) \leftrightarrow (-1, -2, 3)$

- $(\mathbf{a} \times \mathbf{b}) + (\mathbf{a} \times \mathbf{c}) \leftrightarrow (-1, -2, 3)$
- 2. *Perform the indicated vector operations.*
 - a. Write the components of the given vectors (a,b,c) n terms of the base vectors $(\mathbf{\hat{x}}_1, \mathbf{\hat{x}}_2, \mathbf{\hat{x}}_3)$ provided:

$$\begin{split} & \mathbf{\hat{x}}_{1}^{'} \leftrightarrow (0.866, \ 0.500, \ 0.000) \\ & \mathbf{\hat{x}}_{2}^{'} \leftrightarrow (-0.500, \ 0.866, \ 0.000) \\ & \mathbf{\hat{x}}_{3}^{'} \leftrightarrow (0.000, \ 0.000, \ 1.000) \end{split}$$

where the components of these base vectors are expressed in the original coordinate system as follows:

 $\mathbf{a} \leftrightarrow (1, 1, 1)$ $\mathbf{b} \leftrightarrow (1, 2, 3)$ $\mathbf{c} \leftrightarrow (-1, 1, -1)$

or

$$\mathbf{a} = \mathbf{\hat{x}}_1 + \mathbf{\hat{x}}_2 + \mathbf{\hat{x}}_3$$
 $\mathbf{b} = \mathbf{\hat{x}}_1 + 2\mathbf{\hat{x}}_2 + 3\mathbf{\hat{x}}_3$ $\mathbf{c} = -\mathbf{\hat{x}}_1 + \mathbf{\hat{x}}_2 - \mathbf{\hat{x}}_3$

b. Perform the following operations using the components of **a**, **b**, **c** expressed in the new (primed) basis:

a∙ b	a (b×c)	a×(b+c)
a×b	(a×b) (a×c)	$(\mathbf{a} \times \mathbf{b}) + (\mathbf{a} \times \mathbf{c})$
a+b		

- c. Construct the rotation matrix [R] to transform components from the original coordinate system to the primed coordinate system. Is [R] orthogonal? Find the inverse of [R] in order to transform components expressed in the primed coordinate system back to the original, unprimed coordinate system.
- d. Transform the components of the results found in Part b. to the unprimed coordinate system and compare the results with the equivalent operations carried out in Part a. (using components expressed in the original coordinate system).

SOLUTION:

a. $\begin{array}{lll}
\hat{\mathbf{x}}_{1}' \leftrightarrow (0.866, 0.500, 0.000) & \hat{\mathbf{x}}_{1}' = 0.866 \, \hat{\mathbf{x}}_{1} + 0.5 \, \hat{\mathbf{x}}_{2} \\
\hat{\mathbf{x}}_{2}' \leftrightarrow (-0.500, 0.866, 0.000) & \hat{\mathbf{x}}_{2}' = -0.5 \, \hat{\mathbf{x}}_{1} + 0.866 \, \hat{\mathbf{x}}_{2} \\
\hat{\mathbf{x}}_{3}' \leftrightarrow (0.000, 0.000, 1.000) & \hat{\mathbf{x}}_{3}' = \hat{\mathbf{x}}_{3}
\end{array}$ so that, according to Eq. 2.23:

$$\begin{bmatrix} \mathbf{R} \end{bmatrix} = \begin{bmatrix} 0.866 & 0.500 & 0.000 \\ -0.500 & 0.866 & 0.000 \\ 0.000 & 0.000 & 1.000 \end{bmatrix}, \text{ and}$$
$$\begin{bmatrix} \mathbf{a}_1 \\ \mathbf{a}_2 \\ \mathbf{a}_3 \end{bmatrix} = \begin{bmatrix} 0.866 & 0.500 & 0.000 \\ -0.500 & 0.866 & 0.000 \\ 0.000 & 0.000 & 1.000 \end{bmatrix} \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} = \begin{bmatrix} 1.366 \\ 0.366 \\ 1.000 \end{bmatrix}$$
$$\begin{bmatrix} \mathbf{b}_1 \\ \mathbf{b}_2 \\ \mathbf{b}_3 \end{bmatrix} = \begin{bmatrix} 0.866 & 0.500 & 0.000 \\ -0.500 & 0.866 & 0.000 \\ 0.000 & 0.000 & 1.000 \end{bmatrix} \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix} = \begin{bmatrix} 1.866 \\ 1.232 \\ 3.000 \end{bmatrix}$$
$$\begin{bmatrix} \mathbf{c}_1 \\ \mathbf{c}_2 \\ \mathbf{c}_3 \end{bmatrix} = \begin{bmatrix} 0.866 & 0.500 & 0.000 \\ -0.500 & 0.866 & 0.000 \\ 0.000 & 0.000 & 1.000 \end{bmatrix} \begin{bmatrix} -1 \\ 1 \\ -1 \end{bmatrix} = \begin{bmatrix} 0.366 \\ 1.366 \\ -1.000 \end{bmatrix}$$
$$\mathbf{a}' \leftrightarrow (1.366, 0.366, 1.000)$$
$$\mathbf{b}' \leftrightarrow (1.866, 1.232, 3.000)$$

 $\mathbf{c'} \leftrightarrow (-0.366, 1.366, -1.000)$

b.

 $\mathbf{a'} \cdot \mathbf{b'} = 6$ (Note: the prime notation is used here to remind that the required operations were carried out using the components expressed in the primed coordinate system.)

 $\mathbf{a'} \times \mathbf{b'} \leftrightarrow (-0.134, -2.232, 1.000)$

 $\mathbf{a'} + \mathbf{b'} \leftrightarrow (3.232, 1.598, 4.000)$

$$\mathbf{a}' \cdot (\mathbf{b}' \times \mathbf{c}') = -4.000$$

$$(\mathbf{a'} \times \mathbf{b'}) \cdot (\mathbf{a'} \times \mathbf{c'}) = 0.000$$

 $(\mathbf{a} \times \mathbf{b}) + (\mathbf{a} \times \mathbf{c}) = -4.000$

 $(\mathbf{a'} \times \mathbf{b'}) + (\mathbf{a'} \times \mathbf{c'}) \leftrightarrow (-1.866, -1.232, 3)$

c.
$$\begin{bmatrix} R \end{bmatrix} = \begin{bmatrix} 0.866 & 0.500 & 0.000 \\ -0.500 & 0.866 & 0.000 \\ 0.000 & 0.000 & 1.000 \end{bmatrix}, \quad \begin{bmatrix} x' \end{bmatrix} = \begin{bmatrix} R \end{bmatrix} \begin{bmatrix} x \end{bmatrix}$$

$$\begin{bmatrix} \mathbf{R} \end{bmatrix} \begin{bmatrix} \mathbf{R}^{\mathrm{T}} \end{bmatrix} = \begin{bmatrix} 0.866 & 0.500 & 0.000 \\ -0.500 & 0.866 & 0.000 \\ 0.000 & 0.000 & 1.000 \end{bmatrix} \begin{bmatrix} 0.866 & -0.500 & 0.000 \\ 0.500 & 0.866 & 0.000 \\ 0.000 & 0.000 & 1.000 \end{bmatrix} = \begin{bmatrix} 1.00 & 0.00 & 0.00 \\ 0.00 & 1.00 & 0.00 \\ 0.00 & 0.00 & 1.000 \end{bmatrix} = \begin{bmatrix} \mathbf{I} \end{bmatrix}$$

Orthogonal

$$\left[\mathbf{R}\right]^{-1} = \frac{\left[\begin{array}{c} \text{signed} \\ \text{cofactor} \\ \text{matrix} \end{array}\right]^{-1}}{\left|\mathbf{A}\right|} = \left[\begin{array}{c} 0.866 & -0.500 & 0.000 \\ 0.500 & 0.866 & 0.000 \\ 0.000 & 0.000 & 1.000 \end{array}\right] = \left[\mathbf{R}\right]^{\mathrm{T}}$$

d. $\begin{bmatrix} a' \end{bmatrix} = \begin{bmatrix} R \end{bmatrix} \begin{bmatrix} a \end{bmatrix}$, or $\begin{bmatrix} R \end{bmatrix}^{-1} \begin{bmatrix} a' \end{bmatrix} = \begin{bmatrix} a \end{bmatrix}$

(i) $\mathbf{a} \cdot \mathbf{b} = 6$; does not change since it is scalar.

(ii)
$$\mathbf{a} \times \mathbf{b} = \begin{bmatrix} 0.866 & -0.500 & 0.000\\ 0.500 & 0.866 & 0.000\\ 0.000 & 0.000 & 1.000 \end{bmatrix} \begin{bmatrix} -0.134\\ -2.232\\ 1.000 \end{bmatrix} = \begin{bmatrix} 1\\ -2\\ 1 \end{bmatrix} \leftrightarrow (1, -2, 1) \text{ (same result)}$$

(iii)
$$\mathbf{a} + \mathbf{b} = \begin{bmatrix} 0.866 & -0.500 & 0.000 \\ 0.500 & 0.866 & 0.000 \\ 0.000 & 0.000 & 1.000 \end{bmatrix} \begin{bmatrix} 3.232 \\ 1.598 \\ 4.000 \end{bmatrix} = \begin{bmatrix} 2 \\ 3 \\ 4 \end{bmatrix} \iff (2, 3, 4) \text{ (same result)}$$

(iv)
$$\mathbf{a} \cdot (\mathbf{b} \times \mathbf{c}) = \mathbf{a} \cdot (\mathbf{b} \times \mathbf{c}) = -4$$

$$(\mathbf{v}) \quad \mathbf{a} \cdot (\mathbf{b} \times \mathbf{c}) = \mathbf{a} \cdot (\mathbf{b} \times \mathbf{c}) = 0$$

(vi)
$$(\mathbf{a} \times \mathbf{b}) \cdot (\mathbf{a} \times \mathbf{c}) = (\mathbf{a} \times \mathbf{b}) \cdot (\mathbf{a} \times \mathbf{c}) \leftrightarrow (-1, -2, 3)$$

(vii)
$$\mathbf{a} \times (\mathbf{b} + \mathbf{c}) = \mathbf{a} \times (\mathbf{b} + \mathbf{c}) \leftrightarrow (-1, -2, 3)$$

(viii)
$$(\mathbf{a} \times \mathbf{b}) + (\mathbf{a} \times \mathbf{c}) = (\mathbf{a} \times \mathbf{b}) + (\mathbf{a} \times \mathbf{c}) \leftrightarrow (-1, -2, 3)$$

Should have the same results for every case.

- 3. Find the rotation matrices for the following operations:
 - a. Rotation of axes (i.e. component transformation) ${}^{45^{\circ}}$ about $\mathbf{\hat{x}}_3$ in a right-handed sense (counter-clockwise when looking anti-parallel along $\mathbf{\hat{x}}_3$).
 - b. Rotation of a physical vector 45° about $\hat{\mathbf{x}}_3$ in a right-handed sense (i.e. the vector moves

counter-clockwise when looking anti-parallel along $\widehat{\mathbf{x}}_{3}$).

- c. Rotation of axes (i.e. component transformation) 30° about $\hat{\mathbf{x}}_2$ in a right-handed sense (i.e. counter-clockwise when viewed anti-parallel to $\hat{\mathbf{x}}_2$).
- *d.* Rotation of a physical vector 30° about $\hat{\mathbf{x}}_2$ in a right-handed sense (i.e. the vector moves counter-clockwise when looking anti-parallel along $\hat{\mathbf{x}}_2$).

SOLUTION:

a.

$$\begin{bmatrix} \mathbf{R} \end{bmatrix} = \begin{bmatrix} \cos 45^{\circ} & \cos 45^{\circ} & \cos 90^{\circ} \\ \cos 135^{\circ} & \cos 45^{\circ} & \cos 90^{\circ} \\ \cos 90^{\circ} & \cos 90^{\circ} & \cos 0^{\circ} \end{bmatrix} = \begin{bmatrix} \frac{\sqrt{2}}{2} & \frac{\sqrt{2}}{2} & 0 \\ -\frac{\sqrt{2}}{2} & \frac{\sqrt{2}}{2} & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$\begin{bmatrix} R \end{bmatrix} = \begin{bmatrix} \frac{\sqrt{2}}{2} & -\frac{\sqrt{2}}{2} & 0 \\ -\frac{\sqrt{2}}{2} & \frac{\sqrt{2}}{2} & 0 \\ 0 & 0 & 1 \end{bmatrix} \qquad \begin{bmatrix} R \end{bmatrix} = \begin{bmatrix} \frac{\sqrt{3}}{2} & 0 & -\frac{1}{2} \\ 0 & 1 & 0 \\ \frac{1}{2} & 0 & \frac{\sqrt{3}}{2} \end{bmatrix} \qquad \begin{bmatrix} R \end{bmatrix} = \begin{bmatrix} \frac{\sqrt{3}}{2} & 0 & -\frac{1}{2} \\ 0 & 1 & 0 \\ -\frac{1}{2} & 0 & \frac{\sqrt{3}}{2} \end{bmatrix}$$

b.

- 4. Perform the matrix manipulations shown.
- a. Find the determinants and inverses of the following matrices: $\begin{bmatrix} A \end{bmatrix} = \begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 8 & 9 \end{bmatrix} \quad \begin{bmatrix} B \end{bmatrix} = \begin{bmatrix} 7 & 8 & 9 \\ 1 & 2 & 3 \\ 4 & 5 & 6 \end{bmatrix} \quad \begin{bmatrix} C \end{bmatrix} = \begin{bmatrix} 1 & 1 & 1 \\ -1 & 2 & 3 \\ 3 & 1 & -1 \end{bmatrix}$
 - b. Multiply [A][A]⁻¹, [B][B]⁻¹, and [C][C]⁻¹ to verify that the inverse has been correctly obtained.

SOLUTION:

$$\begin{bmatrix} A \end{bmatrix} = \begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 8 & 9 \end{bmatrix} \qquad \begin{bmatrix} B \end{bmatrix} = \begin{bmatrix} 7 & 8 & 9 \\ 1 & 2 & 3 \\ 4 & 5 & 6 \end{bmatrix} \qquad \begin{bmatrix} C \end{bmatrix} = \begin{bmatrix} 1 & 1 & 1 \\ -1 & 2 & 3 \\ 3 & 1 & -1 \end{bmatrix}$$

a.

$$A = 1(5 \cdot 9 - 6 \cdot 8) - 4(2 \cdot 9 - 3 \cdot 8) + 7(2 \cdot 6 - 3 \cdot 5) = 0$$

$$\begin{bmatrix} A \end{bmatrix}^{-1} = \frac{\begin{bmatrix} \text{signed wfactor} \\ \text{matrix} \end{bmatrix}^{T}}{|A|}, \quad \text{since } |A| = 0, \quad [A]^{-1} \text{ cannot be obtained}$$
$$|B| = |A| = 0, \quad [B]^{-1} \to \text{does not exist}$$
$$\begin{bmatrix} C \end{bmatrix}^{-1} = \begin{bmatrix} 1.25 & -0.5 & -0.25 \\ -2 & 1 & 1 \\ 1.75 & -0.5 & -0.75 \end{bmatrix}$$
$$\begin{bmatrix} C \end{bmatrix}^{-1} = \begin{bmatrix} 1.25 & -0.5 & -0.25 \\ -2 & 1 & 1 \\ 1.75 & -0.5 & -0.75 \end{bmatrix}$$
$$\begin{bmatrix} A \end{bmatrix} \begin{bmatrix} A \end{bmatrix}^{-1}, \quad \begin{bmatrix} B \end{bmatrix} \begin{bmatrix} B \end{bmatrix}^{-1}; \text{ not applicable.}$$
$$\begin{bmatrix} C \end{bmatrix} \begin{bmatrix} C \end{bmatrix}^{-1} = \begin{bmatrix} 1 & 1 & 1 \\ -1 & 2 & 3 \\ 3 & 1 & -1 \end{bmatrix} \begin{bmatrix} 1.25 & -0.5 & -0.25 \\ -2 & 1 & 1 \\ 1.75 & -0.5 & -0.75 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

 \therefore Yes, inverse has been correctly obtained.

5. The following sets of basis vectors are presented in a standard Cartesian coordinate system $(\hat{\mathbf{x}}_1, \hat{\mathbf{x}}_2, \hat{\mathbf{x}}_3)$.

Set (1)
$$\begin{cases} \mathbf{\hat{x}}_{1}^{(1)} \leftrightarrow (0.707, 0.707, 0.000) \\ \mathbf{\hat{x}}_{2}^{(1)} \leftrightarrow (-0.500, 0.500, 0.707) \\ \mathbf{\hat{x}}_{3}^{(1)} \leftrightarrow (0.500, -0.500, 0.707) \\ \end{cases}$$
$$\begin{cases} \mathbf{\hat{x}}_{1}^{(2)} \leftrightarrow (0.750, 0.433, 0.500) \\ \mathbf{\hat{x}}_{2}^{(2)} \leftrightarrow (-0.500, 0.866, 0.000) \\ \mathbf{\hat{x}}_{3}^{(2)} \leftrightarrow (-0.433, -0.250, 0.866) \\ \end{cases}$$
$$\begin{cases} \mathbf{\hat{x}}_{1}^{(3)} \leftrightarrow (0.866, 0.500, 0.354) \\ \mathbf{\hat{x}}_{2}^{(3)} \leftrightarrow (0.500, 0.866, 0.354) \\ \mathbf{\hat{x}}_{3}^{(3)} \leftrightarrow (0.000, 0.000, 0.866) \\ \end{cases}$$

- a. Using vector operations, determine which of the basis sets are orthogonal.
- b. Determine the transformation matrices to transform components presented in the original coordinate system $(\hat{\mathbf{x}}_1, \hat{\mathbf{x}}_2, \hat{\mathbf{x}}_3)$ to those in each of the other basis systems.

- c. Which of the transformation matrices in Part b. are orthogonal? Does this agree with Part a?
- d. Find the transformation matrix to transform components provided in coordinate system (1) to components expressed in coordinate system (2). Is the transformation matrix orthogonal?

SOLUTION:

To be orthogonal, the inner product of two vectors should be zero. a.

Set (1)
$$\mathbf{\hat{x}}_{1}^{(1)} \cdot \mathbf{\hat{x}}_{2}^{(1)} = (0.707, 0.707, 0.000) \cdot (-0.5, 0.5, 0.707) = 0.0$$

 $\mathbf{\hat{x}}_{3}^{(1)} \cdot \mathbf{\hat{x}}_{2}^{(1)} = (0.500, -0.500, 0.707) \cdot (-0.500, 0.500, 0.707) = 0.0$
 $\mathbf{\hat{x}}_{1}^{(1)} \cdot \mathbf{\hat{x}}_{3}^{(1)} = (0.707, 0.707, 0.000) \cdot (0.500, -0.500, 0.707) = 0.0$
 \therefore orthogonal

Set (2)
$$\hat{\mathbf{x}}_{1}^{(2)} \cdot \hat{\mathbf{x}}_{2}^{(2)} = \hat{\mathbf{x}}_{2}^{(2)} \cdot \hat{\mathbf{x}}_{3}^{(2)} = \hat{\mathbf{x}}_{1}^{(2)} \cdot \hat{\mathbf{x}}_{3}^{(2)} = 0.0$$
 \therefore orthogonal

Set (3)
$$\hat{\mathbf{x}}_{2}^{(3)} \cdot \hat{\mathbf{x}}_{3}^{(3)} = 0.991$$

 $\hat{\mathbf{x}}_{1}^{(3)} \cdot \hat{\mathbf{x}}_{2}^{(3)} = 0.307$
 $\hat{\mathbf{x}}_{1}^{(3)} \cdot \hat{\mathbf{x}}_{3}^{(3)} = 0.307$ \therefore not orthogonal

$$\mathbf{\hat{x}}_1 \leftrightarrow (1,0,0), \quad \mathbf{\hat{x}}_2 \leftrightarrow (0,1,0), \quad \mathbf{\hat{x}}_3 \leftrightarrow (0,0,0)$$

$$\begin{cases} \mathbf{\hat{x}}_{1}^{(1)} \leftrightarrow (0.707, 0.707, 0.0) = 0.707 (1, 0, 0) + 0.707 (0, 1, 0) + 0.0 (0, 0, 1) \\ \mathbf{\hat{x}}_{2}^{(1)} \leftrightarrow (-0.5, 0.5, 0.707) = -0.5 (1, 0, 0) + 0.5 (0, 1, 0) + 0.707 (0, 0, 1) \\ \mathbf{\hat{x}}_{3}^{(1)} \leftrightarrow (0.5, -0.5, 0.707) = 0.5 (1, 0, 0) + -0.5 (0, 1, 0) + 0.707 (0, 0, 1) \\ \end{cases}$$
Set(1)

1)

In a similar way as shown in Exercise 2.5, we obtain

$$\begin{bmatrix} \mathbf{R}^{(1)} \end{bmatrix} = \begin{bmatrix} 0.707 & 0.707 & 0.000 \\ -0.500 & 0.500 & 0.707 \\ 0.500 & -0.500 & 0.707 \end{bmatrix}$$

Set (1)
$$\begin{bmatrix} \mathbf{R}^{(2)} \end{bmatrix} = \begin{bmatrix} 0.750 & 0.433 & 0.500 \\ -0.500 & 0.866 & 0.000 \\ -0.433 & -0.250 & 0.866 \end{bmatrix}$$

Set (2)
$$\begin{bmatrix} \mathbf{R}^{(3)} \end{bmatrix} = \begin{bmatrix} 0.866 & 0.500 & 0.354 \\ 0.500 & 0.866 & 0.354 \\ 0.000 & 0.000 & 0.866 \end{bmatrix}$$

Set (3)

c.

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$$\begin{bmatrix} \mathbf{R}^{(1)} \end{bmatrix} \begin{bmatrix} \mathbf{R}^{(1)} \end{bmatrix}^{\mathrm{T}} = \begin{bmatrix} 0.707 & -0.5 & 0.5 \\ 0.707 & 0.5 & -0.5 \\ 0.0 & 0.707 & 0.707 \end{bmatrix} \begin{bmatrix} 0.707 & 0.707 & 0.0 \\ -0.5 & 0.5 & 0.707 \\ 0.5 & -0.5 & 0.707 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

Set (1)

Set (1)
$$\begin{bmatrix} 0.0 & 0.707 & 0.707 \end{bmatrix} \begin{bmatrix} 0.5 & -0.5 & 0.707 \end{bmatrix}$$

 \therefore orthogonal
Set (2) $\begin{bmatrix} R^{(2)} \end{bmatrix} \begin{bmatrix} R^{(2)} \end{bmatrix}^{T} = I$ \therefore orthogonal

$$\begin{bmatrix} \mathbf{R}^{(3)} \end{bmatrix} \begin{bmatrix} \mathbf{R}^{(3)} \end{bmatrix}^{\mathrm{T}} = \begin{bmatrix} 0.866 & 0.500 & 0.354 \\ 0.500 & 0.866 & 0.354 \\ 0.000 & 0.000 & 0.866 \end{bmatrix} \begin{bmatrix} 0.866 & 0.500 & 0.000 \\ 0.500 & 0.866 & 0.000 \\ 0.354 & 0.354 & 0.866 \end{bmatrix} = \begin{bmatrix} 1.13 & 0.99 & 0.31 \\ 0.99 & 1.13 & 0.31 \\ 0.31 & 0.31 & 0.75 \end{bmatrix} \neq \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} \mathbf{I} \end{bmatrix} \quad \therefore \text{ not orthogonal}$$

Set (3)

i.e. When the transformation is a pure rotation, the transformation is orthogonal. All of these results agree with Part a.

d.
$$\mathbf{x}^{(1)} = \begin{bmatrix} \mathbf{R}^{(1)} \end{bmatrix} \begin{bmatrix} \mathbf{x} \end{bmatrix}, \quad \mathbf{x}^{(2)} = \begin{bmatrix} \mathbf{R}^{(2)} \end{bmatrix} \begin{bmatrix} \mathbf{x} \end{bmatrix}$$

Thus, $\begin{bmatrix} \mathbf{x}^{(2)} \end{bmatrix} = \begin{bmatrix} \mathbf{R}^{(2)} \end{bmatrix} \begin{bmatrix} \mathbf{R}^{(1)} \end{bmatrix}^{-1} \begin{bmatrix} \mathbf{x}^{(1)} \end{bmatrix}$
Since $\begin{bmatrix} \mathbf{R}^{(1)} \end{bmatrix}$ is orthogonal $\begin{bmatrix} \mathbf{R}^{(1)} \end{bmatrix}^{-1} = \begin{bmatrix} \mathbf{R}^{(1)} \end{bmatrix}^{\mathrm{T}}$
 $\therefore \begin{bmatrix} \mathbf{R}^{(1) \to (2)} \end{bmatrix} = \begin{bmatrix} \mathbf{R}^{(2)} \end{bmatrix} \begin{bmatrix} \mathbf{R}^{(1)} \end{bmatrix}^{\mathrm{T}}$
 $= \begin{bmatrix} 0.750 & 0.433 & 0.500 \\ -0.500 & 0.866 & 0.000 \\ -0.433 & -0.250 & 0.866 \end{bmatrix} \begin{bmatrix} 0.707 & -0.500 & 0.500 \\ 0.707 & 0.500 & -0.500 \\ 0.000 & 0.707 & 0.707 \end{bmatrix}$
 $= \begin{bmatrix} 0.836 & 0.195 & 0.512 \\ 0.259 & 0.683 & -0.683 \\ -0.483 & 0.704 & 0.521 \end{bmatrix}$

Check of orthogonality:

$$\begin{bmatrix} \mathbf{R}' \end{bmatrix} \begin{bmatrix} \mathbf{R}^{(1)} \end{bmatrix}^{\mathrm{T}} = \begin{bmatrix} 0.836 & 0.195 & 0.512 \\ 0.259 & 0.683 & -0.683 \\ -0.483 & 0.704 & 0.521 \end{bmatrix} \begin{bmatrix} 0.836 & 0.259 & -0.483 \\ 0.195 & 0.683 & 0.704 \\ 0.512 & -0.683 & 0.521 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} \mathbf{I} \end{bmatrix}$$

$$\therefore \text{ orthogonal}$$

For those familiar with matrix manipulation, another proof may be written briefly as follows:

$$\begin{bmatrix} \mathbf{R}^{(1)} \end{bmatrix} \begin{bmatrix} \mathbf{R}^{(1)} \end{bmatrix}^{\mathrm{T}} = \begin{bmatrix} \mathbf{R}^{(2)} \end{bmatrix} \begin{bmatrix} \mathbf{R}^{(1)} \end{bmatrix}^{\mathrm{T}} \begin{pmatrix} \begin{bmatrix} \mathbf{R}^{(2)} \end{bmatrix} \begin{bmatrix} \mathbf{R}^{(1)} \end{bmatrix} \end{pmatrix}^{\mathrm{T}}$$
$$= \begin{bmatrix} \mathbf{R}^{(2)} \end{bmatrix} \underbrace{\begin{bmatrix} \mathbf{R}^{(1)} \end{bmatrix}^{\mathrm{T}} \begin{bmatrix} \mathbf{R}^{(1)} \end{bmatrix}}_{\begin{bmatrix} \mathbf{I} \end{bmatrix}} \begin{bmatrix} \mathbf{R}^{(2)} \end{bmatrix}^{\mathrm{T}} = \begin{bmatrix} \mathbf{R}^{(2)} \end{bmatrix} \begin{bmatrix} \mathbf{R}^{(2)} \end{bmatrix}^{\mathrm{T}} = \begin{bmatrix} \mathbf{I} \end{bmatrix}$$

6. Solve the sets of equations presented below by finding the inverse of the coefficient matrix. (Note that Part b will require extension of the inversion formula to matrices of size greater than 3x3.):

 $X_1 + 2X_2 + 3X_3 = 10$ $X_1 + 5X_2 - X_3 = 12$ $X_1 + 3X_2 + X_3 = 14$

а.

$$X_{1} + 2X_{2} + 3X_{3} + 4X_{4} = 10$$

$$X_{1} + 5X_{2} - X_{3} + 14X_{4} = 12$$

$$X_{1} + 3X_{2} + X_{3} + X_{4} = 14$$

$$X_{1} + 4X_{2} - 2X_{3} - 2X_{4} = 16$$

b.

SOLUTION:

a.

$$\begin{bmatrix} \mathbf{K} \end{bmatrix} \begin{bmatrix} \mathbf{X} \end{bmatrix} = \begin{bmatrix} \mathbf{F} \end{bmatrix} \rightarrow \begin{bmatrix} \mathbf{X} \end{bmatrix} = \begin{bmatrix} \mathbf{K} \end{bmatrix}^{-1} \begin{bmatrix} \mathbf{F} \end{bmatrix}$$

$$\begin{bmatrix} \mathbf{K} \end{bmatrix} = \begin{bmatrix} 1 & 2 & 3 \\ 1 & 5 & -1 \\ 1 & 3 & 1 \end{bmatrix}, \qquad \begin{bmatrix} \mathbf{F} \end{bmatrix} = \begin{bmatrix} 10 \\ 12 \\ 14 \end{bmatrix}$$

$$\therefore \begin{bmatrix} \mathbf{K} \end{bmatrix}^{-1} = -\frac{1}{2} \begin{bmatrix} 8 & -2 & -2 \\ 7 & -2 & -1 \\ -17 & 4 & 3 \end{bmatrix}^{\mathrm{T}}$$

Classical adjoint: transpose of the matrix of cofactors.

$$\therefore \begin{bmatrix} X \end{bmatrix} = -\frac{1}{2} \begin{bmatrix} 8 & 7 & -17 \\ -2 & -2 & 4 \\ -2 & -1 & 3 \end{bmatrix} \begin{bmatrix} 10 \\ 12 \\ 14 \end{bmatrix} = \begin{bmatrix} 37 \\ -6 \\ -5 \end{bmatrix}$$
$$\begin{bmatrix} X \end{bmatrix} = \begin{bmatrix} -4.632 \\ 5.789 \\ 2.000 \\ -0.737 \end{bmatrix}$$
In the same way,

b.

- 7. Perform the following operations related to eigenvector eigenvalue problems.
 - a. Find the eigenvalues and the associated eigenvectors for the following matrices:

[1 2]	123	1 -1 2
	2 4 5	-1 2 -3
	356	2 -3 3

- b. Find the transformation matrices which change components expressed in the original coordinate system to ones expressed using the eigenvectors as base vectors. Choose the direction associated with the maximum eigenvalue to be the new $\hat{\mathbf{x}}_1$ ', the second one $\hat{\mathbf{x}}_2$ ', and the third one $\hat{\mathbf{x}}_3$ '.
- c. Treating the columns of the matrices in part a. as vectors, find the equivalent components expressed in the eigenvector bases from Part b.. [i.e. use the transformation matrices found in Part c. to find the new components of the tensors in Part a., expressed in the **principal** coordinate system.]

SOLUTION:

a.
$$|\mathbf{A} - \lambda \mathbf{I}| = 0$$
, $\begin{vmatrix} 1 - \lambda & 2 \\ 3 & 1 - \lambda \end{vmatrix} = (1 - \lambda)^2 - 6 = 1 - 2\lambda + \lambda^2 - 6 = \lambda^2 - 2\lambda - 5 = 0$

Eigenvalues: $\lambda_1 = 1 + \sqrt{6}$, $\lambda_2 = 1 - \sqrt{6}$; Eigenvectors: $\mathbf{p}^{(1)}$, $\mathbf{p}^{(2)}$

(i) For
$$\lambda = 1 + \sqrt{6} = 3.449$$
, $\mathbf{p}^{(1)} \leftrightarrow \begin{bmatrix} x_1^{(1)} \\ x_2^{(1)} \end{bmatrix}$
 $\begin{bmatrix} A - \lambda_1 I \end{bmatrix} \begin{bmatrix} p^{(1)} \end{bmatrix} = \begin{bmatrix} -\sqrt{6} & 2 \\ 3 & -\sqrt{6} \end{bmatrix} \begin{bmatrix} x_1^{(1)} \\ x_2^{(1)} \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$
Let $x_1^{(1)} = 1$, then $-\sqrt{6} + 2x_2^{(1)} = 0$ $\therefore x_2^{(1)} = \frac{\sqrt{6}}{2}$

Normalizing these
$$\left[p^{(1)}\right] \leftrightarrow \frac{1}{\sqrt{\frac{5}{2}}} \left(1, \frac{\sqrt{6}}{2}\right) = (0.632, 0.775)$$

(ii) For
$$\lambda = 1 - \sqrt{6} = -1.449$$

$$\begin{bmatrix} \sqrt{6} & 2 \\ 3 & \sqrt{6} \end{bmatrix} \begin{bmatrix} x_1^{(2)} \\ x_2^{(2)} \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}, \text{ and if } x_1^{(2)} = 1, \text{ then } x_2^{(2)} = \frac{-\sqrt{6}}{2}$$

Normalizing, we get $\mathbf{p}^{(2)} \leftrightarrow (0.632, -0.775)$

 $\mathbf{p}^{(1)} \cdot \mathbf{p}^{(2)} = -0.20 \neq 0$ (Not orthogonal because the Check of orthogonality: original matrix is not symmetric.)

Using the same procedure,

$$\begin{array}{l} \lambda_{1} = 11.345, \quad \mathbf{p}_{1} \leftrightarrow (0.328, 0.591, 0.737) \\ \lambda_{2} = -0.516, \quad \mathbf{p}_{2} \leftrightarrow (0.737, 0.328, -0.591) \\ \lambda_{3} = 0.171, \quad \mathbf{p}_{3} \leftrightarrow (-0.591, 0.737, -0.328) \\ \end{array}$$

$$\begin{array}{l} \left[\begin{array}{c} 1 & -1 & 2 \\ -1 & 2 & -3 \\ 2 & -3 & 3 \end{array} \right], \qquad \lambda_{1} = 6.419, \quad \mathbf{p}_{1} \leftrightarrow (0.374, -0.577, 0.725) \\ \lambda_{2} = -0.387, \quad \mathbf{p}_{2} \leftrightarrow (0.816, 0.577, 0.038) \\ \lambda_{3} = 0.806, \quad \mathbf{p}_{3} \leftrightarrow (-0.441, 0.577, 0.687) \\ \end{array}$$

$$\left[\mathbf{T} \right]^{(1)} = \left[\begin{array}{c} 0.632 & 0.775 \\ 0.632 & -0.775 \end{array} \right]$$

$$\left[\mathbf{T} \right]^{(2)} = \left[\begin{array}{c} 0.328 & 0.591 & 0.737 \\ -0.591 & 0.737 & -0.328 \\ 0.737 & 0.328 & -0.591 \end{array} \right]$$

b.

$$\begin{bmatrix} T \end{bmatrix}^{(2)} = \begin{bmatrix} 0.328 & 0.391 & 0.737 \\ -0.591 & 0.737 & -0.328 \\ 0.737 & 0.328 & -0.591 \end{bmatrix}$$
$$\begin{bmatrix} 0.374 & -0.577 & 0.725 \\ 0.816 & 0.577 & 0.038 \\ -0.041 & 0.577 & 0.687 \end{bmatrix}$$

Suppose x, y, z are orthogonal eigenvectors of A where eigenvalues are λ_1, λ_2 and λ_3 , c. respectively, let

 $\begin{bmatrix} L \end{bmatrix} = \begin{bmatrix} x_1 & y_1 & z_1 \\ x_2 & y_2 & z_2 \\ x_3 & y_3 & z_3 \end{bmatrix}, \quad \text{where} \quad \mathbf{x} \leftrightarrow \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} \quad \mathbf{y} \leftrightarrow \begin{bmatrix} y_1 \\ y_2 \\ y_3 \end{bmatrix} \quad \mathbf{z} \leftrightarrow \begin{bmatrix} z_1 \\ z_2 \\ z_3 \end{bmatrix}$ $\begin{bmatrix} A \end{bmatrix} \begin{bmatrix} x \end{bmatrix} = \lambda_1 \begin{bmatrix} x \end{bmatrix}$ $\begin{bmatrix} A \end{bmatrix} \begin{bmatrix} y \end{bmatrix} = \lambda_2 \begin{bmatrix} y \end{bmatrix}$ Then from 2.35a, $\begin{bmatrix} A \end{bmatrix} \begin{bmatrix} z \end{bmatrix} = \lambda_3 \begin{bmatrix} z \end{bmatrix}$ $\rightarrow [A][L] = [D][L] \qquad \qquad \therefore \quad [D] = [L]^{T}[A][L]$

Here $[L] = [T]^T$, and $[D] = [T] [A] [T]^T$

We will obtain [D]; the diagonal matrix whose diagonal components are eigenvalues. For example,

$$\begin{bmatrix} D \end{bmatrix} = \begin{bmatrix} 0.328 & 0.591 & 0.737 \\ 0.737 & 0.328 & -0.591 \\ -0.591 & 0.737 & -0.328 \end{bmatrix} \begin{bmatrix} 1 & 2 & 3 \\ 2 & 4 & 5 \\ 3 & 5 & 6 \end{bmatrix} \begin{bmatrix} 0.328 & 0.737 & -0.591 \\ 0.591 & 0.328 & 0.737 \\ 0.737 & -0.591 & -0.328 \end{bmatrix} = \begin{bmatrix} 11.345 & 0 & 0 \\ 0 & -0.516 & 0 \\ 0 & 0 & 0.171 \end{bmatrix}$$

8. Find the new components of the tensors provided below if the coordinate system change corresponds to a rotation of 30° about $\hat{\mathbf{x}}_3$:

$$\begin{bmatrix} \mathbf{R} \end{bmatrix} = \begin{bmatrix} 0.866 & 0.500 & 0.000 \\ -0.500 & 0.866 & 0.000 \\ 0.000 & 0.000 & 1.000 \end{bmatrix}$$

$$\mathbf{T}_{1} \leftrightarrow \begin{bmatrix} 1 & 2 & 3 \\ 2 & 4 & 5 \\ 3 & 5 & 6 \end{bmatrix} = \begin{bmatrix} T_{1} \end{bmatrix}, \quad \mathbf{T}_{2} \leftrightarrow \begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 8 & 9 \end{bmatrix} = \begin{bmatrix} T_{2} \end{bmatrix}$$

SOLUTION:

$$\begin{bmatrix} \mathbf{T} \end{bmatrix}' = \begin{bmatrix} \mathbf{R} \end{bmatrix} \begin{bmatrix} \mathbf{T} \end{bmatrix} \begin{bmatrix} \mathbf{R} \end{bmatrix}^{\mathrm{T}}, \quad \begin{bmatrix} \mathbf{T}_{1} \end{bmatrix}' = \begin{bmatrix} \mathbf{R} \end{bmatrix} \begin{bmatrix} \mathbf{T}_{1} \end{bmatrix} \begin{bmatrix} \mathbf{R} \end{bmatrix}^{\mathrm{T}}, \quad \begin{bmatrix} \mathbf{T}_{2} \end{bmatrix}' = \begin{bmatrix} \mathbf{R} \end{bmatrix} \begin{bmatrix} \mathbf{T} \end{bmatrix} \begin{bmatrix} \mathbf{R}_{2} \end{bmatrix}^{\mathrm{T}}$$
$$\begin{bmatrix} \mathbf{T}_{1} \end{bmatrix}' = \begin{bmatrix} 0.866 & 0.500 & 0.000 \\ -0.500 & 0.866 & 0.000 \\ 0.000 & 0.000 & 1.000 \end{bmatrix} \begin{bmatrix} 1 & 2 & 3 \\ 2 & 4 & 5 \\ 3 & 5 & 6 \end{bmatrix} \begin{bmatrix} 0.866 & -0.500 & 0.000 \\ 0.500 & 0.866 & 0.000 \\ 0.000 & 0.000 & 1.000 \end{bmatrix} = \begin{bmatrix} 3.482 & 2.299 & 5.098 \\ 2.299 & 1.518 & 2.830 \\ 5.098 & 2.830 & 6.000 \\ 5.098 & 2.830 & 6.000 \\ 10.062 & 3.428 & 9.000 \end{bmatrix}$$

9. In calculating contact conditions at an interface, it is often necessary to find the unit vector which represents the projection of a given vector (usually the displacement of a material point) onto a plane tangent to the interface. If the normal to the tangent plane is denoted $\hat{\mathbf{n}}$ and the arbitrary vector is \mathbf{a} , find $\hat{\mathbf{t}}$, the unit tangent vector corresponding to the material displacement. [Express the result in terms of \mathbf{a} , $\hat{\mathbf{n}}$, and simple vector operations.]

SOLUTION:

One possibility is based on the use of vector addition and the dot product:

 $\mathbf{\hat{t}} = \frac{\mathbf{a} - (\mathbf{a} \cdot \mathbf{n}) \mathbf{\hat{n}}}{|\mathbf{a} - (\mathbf{a} \cdot \mathbf{n}) \mathbf{n}|} \quad (\text{since } \mathbf{\hat{t}} \text{ is a unit vector})$

Alternatively, one may use the cross product to accomplish the same thing by first defining a unit vector $\hat{\mathbf{q}}$, which is orthogonal to $\hat{\mathbf{n}}$, $\hat{\mathbf{a}}$, and $\hat{\mathbf{t}}$:

$$\mathbf{\hat{q}} = \frac{\mathbf{a} \times \mathbf{\hat{n}}}{|\mathbf{a} \times \mathbf{\hat{n}}|}, \mathbf{\hat{t}} = \mathbf{\hat{n}} \times \mathbf{\hat{q}} = \frac{\mathbf{\hat{n}} \times (\mathbf{a} \times \mathbf{\hat{n}})}{|\mathbf{a} \times \mathbf{\hat{n}}|}$$

B. DEPTH PROBLEMS

- 10. Perform the following operations related to component transformations.
 - a. Find the transformation for components from basis set (2) to basis set (3), in Problem 5, above.
 - b. Find the inverse transformation, that is, one that expresses components in basis set (2) if they are given in basis set (3).
 - c. Using the approach shown in Exercise 2.5, verify that transformation matrices found in Parts a. and b. do, indeed, perform the indicated transformations.
 - *d.* Show the matrix form of the tensor transformation for components given in basis set (2) to those in basis set (3).

SOLUTION:

 $\mathbf{\hat{x}}_{i}^{(2)} \rightarrow \mathbf{\hat{x}}_{i}^{(3)}$

a.

$$\begin{aligned} \mathbf{\hat{x}}^{(2)} &= \left[\mathbf{R}^{(2)} \right] \left[\mathbf{\hat{x}} \right], \quad \left[\mathbf{\hat{x}} \right] = \left[\mathbf{R}^{(2)} \right]^{\mathrm{T}} \left[\mathbf{\hat{x}}^{2} \right] \\ \mathbf{\hat{x}}^{(3)} &= \left[\mathbf{R}^{(3)} \right] \left[\mathbf{\hat{x}} \right], \quad \left[\mathbf{\hat{x}}^{3} \right] = \left[\mathbf{R}^{(3)} \right] \left[\mathbf{R}^{(2)} \right]^{\mathrm{T}} \left[\mathbf{\hat{x}}^{(2)} \right] \\ \therefore \left[\mathbf{R}^{(2) \to (3)} \right] &= \left[\mathbf{R}^{(3)} \right] \left[\mathbf{R}^{(2)} \right]^{\mathrm{T}} = \begin{bmatrix} 0.866 & 0.500 & 0.354 \\ 0.500 & 0.866 & 0.354 \\ 0.000 & 0.000 & 0.866 \end{bmatrix} \begin{bmatrix} 0.750 & -0.500 - 0.433 \\ 0.433 & 0.866 - 0.250 \\ 0.500 & 0.000 & 0.866 \end{bmatrix} \\ &= \begin{bmatrix} 1.043 & 0.000 & -0.193 \\ 0.927 & 0.500 & -0.126 \\ 0.433 & 0.000 & 0.750 \end{bmatrix} \end{aligned}$$

Since the inverse of $\begin{bmatrix} R^{(3)} \end{bmatrix}$ is not used in this transformation, its non-orthogonality is not an issue. However, in part b this requires inverting (rather than transposing) a matrix.

b. Recall from Part a:
$$\left[R^{(2)\rightarrow(3)}\right] = \left[R^{(3)}\right] \left[R^{(2)}\right]^{T}$$

$$\begin{bmatrix} \mathbf{R}^{(2)\to(3)} \end{bmatrix}^{-1} = \left(\begin{bmatrix} \mathbf{R}^{(3)} \end{bmatrix} \begin{bmatrix} \mathbf{R}^{(2)} \end{bmatrix}^{\mathrm{T}} \right)^{-1} = \begin{bmatrix} \mathbf{R}^{(2)} \end{bmatrix} \begin{bmatrix} \mathbf{R}^{(3)} \end{bmatrix}^{-1} = \begin{bmatrix} 0.866 & 0.000 & 0.223 \\ -1.732 & 2.000 & -0.110 \\ -0.500 & 0.000 & 1.204 \end{bmatrix}$$

Where we have used the relationships:
$$\begin{bmatrix} \mathbf{R}^{(2)} \end{bmatrix}^{-1} = \begin{bmatrix} \mathbf{R}^{(2)} \end{bmatrix}^{\mathrm{T}} \text{ and } \left(\begin{bmatrix} \mathbf{A} \end{bmatrix} \begin{bmatrix} \mathbf{B} \end{bmatrix} \right)^{-1} = \begin{bmatrix} \mathbf{B} \end{bmatrix}^{-1} \begin{bmatrix} \mathbf{A} \end{bmatrix}^{-1}$$

c. Let's verify the transformations by considering three vectors, namely those of the original basis set $\hat{\mathbf{x}}_{i}^{(0)}$ (let's label them $\hat{\mathbf{x}}, \hat{\mathbf{y}}, \hat{\mathbf{x}}$ here to simplify the notation). We form the matrix $\begin{bmatrix} A \end{bmatrix}$ (corresponding to the tensor **A** composed of the three vectors) in the usual way, by putting the components of each basis vector into a column. Since we are considering the components of the basis set, $\begin{bmatrix} A \end{bmatrix}$ is the identity matrix:

$$\mathbf{A} \leftrightarrow \begin{bmatrix} \mathbf{A}^{(0)} \end{bmatrix} = \begin{bmatrix} \mathbf{x}_1^{(0)} \ \mathbf{y}_1^{(0)} \ \mathbf{z}_1^{(0)} \\ \mathbf{x}_2^{(0)} \ \mathbf{y}_2^{(0)} \ \mathbf{z}_2^{(0)} \\ \mathbf{x}_3^{(0)} \ \mathbf{y}_3^{(0)} \ \mathbf{z}_3^{(0)} \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} \mathbf{I} \end{bmatrix}$$

We then find the coordinates of these three vectors expressed in the $\hat{\mathbf{x}}^{(2)}$ and $\hat{\mathbf{x}}^{(3)}$ basis sets:

$$\mathbf{A} \leftrightarrow \begin{bmatrix} \mathbf{A}^{(2)} \end{bmatrix} = \begin{bmatrix} \mathbf{R}^{(2)} \end{bmatrix} \begin{bmatrix} \mathbf{I} \end{bmatrix} = \begin{bmatrix} 0.750 & 0.433 & 0.500 \\ -0.500 & 0.866 & 0.000 \\ -0.433 & -0.250 & 0.866 \end{bmatrix}$$
(in basis set 2)
$$\mathbf{A} \leftrightarrow \begin{bmatrix} \mathbf{A}^{(3)} \end{bmatrix} = \begin{bmatrix} \mathbf{R}^{(3)} \end{bmatrix} \begin{bmatrix} \mathbf{I} \end{bmatrix} = \begin{bmatrix} 0.866 & 0.500 & 0.354 \\ 0.500 & 0.866 & 0.354 \\ 0.000 & 0.000 & 0.866 \end{bmatrix}$$
(in basis set 3)

Now, our transformation matrix $\begin{bmatrix} R^{(2)\to(3)} \end{bmatrix}$ must transform the components of any vector expressed in $\hat{\mathbf{x}}_{i}^{(2)}$ to components expressed in $\hat{\mathbf{x}}_{i}^{(3)}$:

$$\begin{bmatrix} A^{(3)} \end{bmatrix} = \begin{bmatrix} R^{(2) \to (3)} \end{bmatrix} \begin{bmatrix} A^{(2)} \end{bmatrix} = \begin{bmatrix} 1.043 & 0.000 & -0.193 \\ 0.927 & 0.500 & -0.126 \\ 0.433 & 0.000 & 0.750 \end{bmatrix} \begin{bmatrix} 0.750 & 0.433 & 0.500 \\ -0.500 & 0.866 & 0.000 \\ -0.433 & -0.250 & 0.866 \end{bmatrix} \\ = \begin{bmatrix} 0.866 & 0.500 & 0.354 \\ 0.500 & 0.866 & 0.354 \\ 0.000 & 0.000 & 0.866 \end{bmatrix} = \begin{bmatrix} A^{(3)} \end{bmatrix}$$

Comparison of $\begin{bmatrix} A^{(3)} \end{bmatrix}$ obtained here with $\begin{bmatrix} A^{(3)} \end{bmatrix}$ above shows that the transformation matrix $\begin{bmatrix} R^{(2)\to(3)} \end{bmatrix}$ performs its intended function. $\begin{bmatrix} R^{(3)\to(2)} \end{bmatrix}$, the inverse of $\begin{bmatrix} R^{(2)\to(3)} \end{bmatrix}$ may be verified in the same manner.

- d. Shown in a.
- 11. Find the rotation matrix for the double rotation of coordinate axes: rotate 90° about $\mathbf{\hat{x}}_{1}$, and then 90° about $\mathbf{\hat{x}}_{3}$.
- SOLUTION:

$$\begin{bmatrix} \mathbf{R}_{1} \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & -1 & 0 \end{bmatrix}$$
$$\begin{bmatrix} \mathbf{R}_{2} \end{bmatrix} = \begin{bmatrix} 0 & 1 & 0 \\ -1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$\begin{bmatrix} \mathbf{R} \end{bmatrix} = \begin{bmatrix} \mathbf{R}_2 \end{bmatrix} \begin{bmatrix} \mathbf{R}_1 \end{bmatrix} = \begin{bmatrix} 0 & 1 & 0 \\ -1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & -1 & 0 \end{bmatrix} \begin{bmatrix} 0 & 0 & 1 \\ -1 & 0 & 0 \\ 0 & -1 & 0 \end{bmatrix}$$
$$\begin{bmatrix} \mathbf{R} \end{bmatrix}_{1-2} = \begin{bmatrix} 0 & 0 & 1 \\ -1 & 0 & 0 \\ 0 & -1 & 0 \end{bmatrix}$$

12. A cylindrical coordinate system is one which rotates according to the coordinates of the point in question. Typically, r, θ , z represent the coordinates of a point, with the base vectors given as $\hat{\mathbf{r}}, \hat{\theta}, \hat{\mathbf{z}}$, for example. The figure at the right shows such a coordinate system and a superimposed Cartesian coordinate system which coincides when $\theta=0$.



- a. Find the transformation matrix to change components expressed in $\hat{\mathbf{x}}_1, \hat{\mathbf{x}}_2, \hat{\mathbf{x}}_3$ to ones expressed in $\hat{\mathbf{r}}, \hat{\mathbf{\theta}}, \hat{\mathbf{z}}$.
- b. Find the cylindrical components of the following vectors expressed in the $\hat{\mathbf{x}}_1, \hat{\mathbf{x}}_2, \hat{\mathbf{x}}_3$ system:

0.000,	0.000
0.500,	0.000
1.000,	0.000
1.000,	1.000
1.000,	0.000
1.000,	1.000
	0.000, 0.500, 1.000, 1.000, 1.000, 1.000,

- c. Find the magnitudes of the vectors given in Part b., first using the Cartesian coordinate system components, then using the cylindrical coordinate system. How is the magnitude of a vector computed in cylindrical coordinates?
- d. Is [R] orthogonal for this transformation? Physically, why or why not?

SOLUTION:

a. $\begin{bmatrix} \mathbf{r} \\ \mathbf{\theta} \\ \mathbf{z} \end{bmatrix} = \begin{bmatrix} \cos \theta & \sin \theta & 0 \\ -\sin \theta & \cos \theta & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} \mathbf{x}_1 \\ \mathbf{x}_2 \\ \mathbf{x}_3 \end{bmatrix}$ from geometry

b. a ↔ (10.000, 0.000, 0.000)

$$\begin{bmatrix} \mathbf{a} \end{bmatrix} = \begin{bmatrix} \mathbf{R} \end{bmatrix} \begin{bmatrix} \mathbf{x} \end{bmatrix}$$
$$\begin{bmatrix} a_1 \\ a_2 \\ a_3 \end{bmatrix} = \begin{bmatrix} \cos \theta & \sin \theta & 0 \\ -\sin \theta & \cos \theta & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 10 \\ 0 \\ 0 \end{bmatrix} = \begin{bmatrix} 10 \cos \theta \\ -10 \sin \theta \\ 0 \end{bmatrix}$$

In a similar way,

$$\begin{bmatrix} b \end{bmatrix} = \begin{bmatrix} 0.866 \cos \theta + 0.5 \sin \theta \\ -0.866 \sin \theta + 0.5 \cos \theta \\ 0 \end{bmatrix} \begin{bmatrix} c \end{bmatrix} = \begin{bmatrix} \cos \theta + \sin \theta \\ -\sin \theta + \cos \theta \\ 0 \end{bmatrix}$$
$$\begin{bmatrix} c \end{bmatrix} = \begin{bmatrix} \cos \theta + \sin \theta \\ -\sin \theta + \cos \theta \\ 1 \end{bmatrix} \begin{bmatrix} e \end{bmatrix} = \begin{bmatrix} \sin \theta \\ \cos \theta \\ 0 \end{bmatrix} \begin{bmatrix} f \end{bmatrix} = \begin{bmatrix} -\cos \theta + \sin \theta \\ \sin \theta + \cos \theta \\ 1 \end{bmatrix}$$

c.

Cartesian Coordinate	Cylindrical Coordinate
$ \mathbf{a} = \sqrt{10^2 + 0^2 + 0^2} = 10$	$ \mathbf{a} = \sqrt{10^2 \cos^2 \theta + 10^2 \sin^2 \theta} = 10$
Likewise, should have the same r	results.

$$\begin{bmatrix} \mathbf{R} \end{bmatrix}^{\mathrm{T}} = \begin{bmatrix} \cos \theta & \sin \theta & 0 \\ -\sin \theta & \cos \theta & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} \cos \theta & -\sin \theta & 0 \\ \sin \theta & \cos \theta & 0 \\ 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} \mathbf{I} \end{bmatrix}$$

d.

The basis sets of each system are mutually orthogonal.

13. Perform the indicated operations related to equation solving.

- a. Solve the equations given in Problem 6 by using Gaussian reduction instead of by finding the inverse. Which to you prefer for large matrices?
- b. Given the solutions obtained in Part a., find the inverse of the coefficient matrix.
- c. For larger sets of equations, why is it easier to solve by a reduction method?

SOLUTION:

a.
$$\begin{bmatrix} 1 & 2 & 3 \\ 1 & 5 & -1 \\ 1 & 3 & 1 \end{bmatrix} \begin{bmatrix} 10 \\ 12 \\ 14 \end{bmatrix} \rightarrow \begin{bmatrix} 1 & 2 & 3 \\ 0 & 3 & -4 \\ 0 & 1 & -2 \end{bmatrix} \begin{bmatrix} 10 \\ 2 \\ 4 \end{bmatrix} \rightarrow \begin{bmatrix} 1 & 2 & 3 \\ 0 & 3 & -4 \\ 0 & -\frac{2}{3} \end{bmatrix} \begin{bmatrix} 10 \\ 2 \\ \frac{10}{3} \end{bmatrix} \qquad \begin{array}{c} x_3 = -5 \\ \therefore & x_2 = -6 \\ x_1 = 37 \end{array}$$

Similarly, we should get the same results for the second set. The Gassian reduction method

is much simpler for large matrices because it operates row-by-row and it is not necessary to keep track of complex expressions.

- b. Solutions given in Problem 6.
- c. For large sets of equations, it is much more complicated to compute the inverse of a matrix, whereas the reduction method does not involve inverse computation. Less computation is required in a reduction method.

CHAPTER 3 - PROBLEM SOLUTIONS

PROFICIENCY PROBLEMS A.

Calculate the 3-D stress tensor components for the rectangular material shown in the figure, first 1. in the coordinate system $\hat{\mathbf{x}}_1$, $\hat{\mathbf{x}}_2$, and $\hat{\mathbf{x}}_3$ and then in the coordinate system $\hat{\mathbf{x}}_1'$, $\hat{\mathbf{x}}_2'$, $\hat{\mathbf{x}}_3'$.



The angle between $\hat{\mathbf{x}}_1$ and $\hat{\mathbf{x}}_1$ is 30°, and the $\hat{\mathbf{x}}_3$ and $\hat{\mathbf{x}}_3$ axes are parallel.

SOLUTION:

Before doing the problem formally using the known tensor transformations, let's approach it from a physical and geometrical standpoint. Because of the equilibrium, we know that the force transmitted throught the cross-sectional area (1mm²) normal to $\hat{\mathbf{x}}_1$ is 200N, and the stress vector operating on that same plane is $S_1 \leftrightarrow (200, 0, 0) \frac{N}{mm^2}$. The other two planes, normal to $\hat{\mathbf{x}}_2$ and $\hat{\mathbf{x}}_3$, contain the force vector and thus have no associated stress vectors:

 $\mathbf{S}_2 \leftrightarrow (0, 0, 0)$ $\mathbf{S}_3 \leftrightarrow (0,0,0)$

Therefore, the stress tensor in the $\hat{\mathbf{x}}_i$ coordinate system may be written:

$\sigma\leftrightarrow$	200 0 0	$\begin{array}{c} 0 \\ 0 \\ 0 \end{array}$	$\begin{array}{c} 0\\ 0\\ 0\\ 0\end{array}$	•
-------------------------	---------------	--	---	---

The situation in the $\hat{\mathbf{x}}_{i}$ coordinate system may be approached similarly by first considering the plane normal to $\hat{\mathbf{x}}_1$ which passes through the rod. The entire 200N of force must be transmitted through this area, which is now larger because of the incline:

$$A_1' = \frac{1 \text{ mm}^2}{\cos 30^\circ} = 1.155 \text{ mm}^2$$

The corresponding stress vector thus has a magnitude of $|\mathbf{S}_1'| = \frac{200 \text{ N}}{1 \text{ mm}^2} \cos 30^\circ = \frac{200 \text{ N}}{1.155 \text{ mm}^2} = 173.2 \text{ N/mm}^2$,

To find the components of this vector along the $\hat{\mathbf{x}}_1$ and $\hat{\mathbf{x}}_2$ vectors, we simply resolve this value:

 $\mathbf{S}_{1} \leftrightarrow (173.2 \cos 30, \ 173.2 \sin 30, \ 0) = (150, 86.6, 0) \frac{N}{mm^{2}}$

To find the stress vector operating on the plane normal to $\hat{\mathbf{x}}_2$, we follow the same procedure:

$$A_{2}' = \frac{1 \text{mm}^{2}}{\sin 30^{\circ}} = 2 \text{ mm}^{2}, \text{ so } |S_{2}'| = 100 \text{ N/mm}^{2},$$

and the associated stress vector components are

 $S_2' \leftrightarrow (100 \cos 30, 100 \sin 30, 0) = (86.6 \text{ N/mm}^2, 50 \text{ N/mm}^2, 0)$

And the entire stress tensor in
$$\hat{\mathbf{x}}_1$$
 is

$$\begin{bmatrix} \sigma \end{bmatrix} = \begin{bmatrix} s_{1'} & s_{2'} & s_{3'} \\ 150 & 86.6 & 0 \\ 86.6 & 50 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

It is much easier and less error-prone to use the known tensor transformation properties to solve the problem, as follows:

Let σ be the stress tensor in the material and t the stress vector, then we have in general: $\sigma \cdot n = t = \frac{F}{a}$

For
$$\mathbf{\hat{n}} = \mathbf{\hat{x}}_1$$
, we get: $\mathbf{t}_1 = \mathbf{\sigma} \cdot \mathbf{\hat{x}}_1 = \frac{\mathbf{F}}{\mathbf{a}}$, so that $\begin{bmatrix} \mathbf{t}_1 \end{bmatrix} = \begin{bmatrix} 200 & 0 & 0\\ 0 & 0 & 0\\ 0 & 0 & 0 \end{bmatrix}$ MPa

Similarly, $\mathbf{t}_2 = \boldsymbol{\sigma} \cdot \mathbf{\hat{x}}_2$, $\mathbf{t}_3 = \boldsymbol{\sigma} \cdot \mathbf{\hat{x}}_3$, and we can conclude that the stress tensor is $\begin{bmatrix} \boldsymbol{\sigma} \end{bmatrix} = \begin{bmatrix} 200 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} MPa$

In order to transform these components to those corresponding to the \mathbf{x}_1 coordinate system, we first find the rotation matrix:

$$\begin{bmatrix} \mathbf{x}' \end{bmatrix} = \begin{bmatrix} \mathbf{R} \end{bmatrix} \begin{bmatrix} \mathbf{x} \end{bmatrix} \implies \begin{bmatrix} \widehat{\mathbf{x}}_1 \\ \widehat{\mathbf{x}}_2 \\ \widehat{\mathbf{x}}_3 \end{bmatrix} = \begin{bmatrix} \frac{\sqrt{3}}{2} & -\frac{1}{2} & 0 \\ \frac{1}{2} & \frac{\sqrt{3}}{2} & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} \widehat{\mathbf{x}}_1 \\ \widehat{\mathbf{x}}_2 \\ \widehat{\mathbf{x}}_3 \end{bmatrix}$$

and then apply the transformation rule for a second-ranked tensor: $\begin{bmatrix} - & 1 \end{bmatrix} \begin{bmatrix} - & 1 \end{bmatrix} \begin{bmatrix} - & -1 \end{bmatrix}$

$$\begin{bmatrix} \boldsymbol{\sigma} \end{bmatrix} = \begin{bmatrix} \mathbf{R} \end{bmatrix} \begin{bmatrix} \boldsymbol{\sigma} \end{bmatrix} \begin{bmatrix} \mathbf{R} \end{bmatrix}^{\mathrm{T}} \implies \begin{bmatrix} \frac{\sqrt{3}}{2} & -\frac{1}{2} & 0 \\ \frac{1}{2} & \frac{\sqrt{3}}{2} & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 200 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} \frac{\sqrt{3}}{2} & \frac{1}{2} & 0 \\ -\frac{1}{2} & \frac{\sqrt{3}}{2} & 0 \\ 0 & 0 & 1 \end{bmatrix}$$
$$= \begin{bmatrix} 100\sqrt{3} & 0 & 0 \\ 100 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} \frac{\sqrt{3}}{2} & \frac{1}{2} & 0 \\ -\frac{1}{2} & \frac{\sqrt{3}}{2} & 0 \\ 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} 150 & 50\sqrt{3} & 0 \\ 50\sqrt{3} & 50 & 0 \\ 0 & 0 & 0 \end{bmatrix} = \begin{bmatrix} 150 & 86.6 & 0 \\ 86.6 & 50 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$
[MPa]

2. Given the stress tensor which appears below, find the stress vector acting on planes normal to the unit vectors $\hat{\mathbf{n}}$, $\hat{\mathbf{m}}$, and $\hat{\mathbf{p}}$, also given.

[1 2 2]	$\mathbf{\hat{n}} \leftrightarrow \frac{1}{\sqrt{3}}(1,1,1)$
$\sigma \leftrightarrow \begin{bmatrix} 1 & 2 & 5 \\ 2 & 2 & 4 \\ 3 & 4 & 3 \end{bmatrix}$	$\widehat{\mathbf{m}} \leftrightarrow \frac{1}{\sqrt{6}}(1,2,1)$
- <i>3</i> - <i>3</i>	$\mathbf{\hat{p}} \leftrightarrow \frac{1}{\sqrt{2}}(1,1,0)$

SOLUTION: $[t] = [\sigma] [n]$

$$\begin{bmatrix} \mathbf{t} \end{bmatrix}^{\mathbf{\hat{m}}} = \begin{bmatrix} 1 & 2 & 3 \\ 2 & 2 & 4 \\ 3 & 4 & 3 \end{bmatrix} \begin{bmatrix} \frac{1}{73} \\ \frac{1}{73} \\ \frac{1}{73} \\ \frac{1}{73} \\ \frac{1}{73} \end{bmatrix} = \begin{bmatrix} 3.464 \\ 4.619 \\ 5.774 \end{bmatrix}$$
$$\begin{bmatrix} \mathbf{t} \end{bmatrix}^{\mathbf{\hat{m}}} = \begin{bmatrix} 1 & 2 & 3 \\ 2 & 2 & 4 \\ 3 & 4 & 3 \end{bmatrix} \begin{bmatrix} \frac{1}{76} \\ \frac{2}{76} \\ \frac{1}{76} \\ \frac{1}{76} \end{bmatrix} = \begin{bmatrix} \frac{8}{76} \\ \frac{10}{76} \\ \frac{15}{76} \end{bmatrix} = \begin{bmatrix} 3.266 \\ 4.082 \\ 5.715 \end{bmatrix}$$
$$\begin{bmatrix} \mathbf{t} \end{bmatrix}^{\mathbf{\hat{m}}} = \begin{bmatrix} 1 & 2 & 3 \\ 2 & 2 & 4 \\ 3 & 4 & 3 \end{bmatrix} \begin{bmatrix} \frac{1}{76} \\ \frac{1}{76} \\ \frac{1}{76} \\ \frac{1}{76} \end{bmatrix} = \begin{bmatrix} \frac{3}{72} \\ \frac{4}{72} \\ \frac{7}{72} \end{bmatrix} = \begin{bmatrix} 2.121 \\ 2.828 \\ 4.950 \end{bmatrix}$$

3. Find the principal stresses, the principal directions, and the rotation matrix for transforming coordinates to the principal coordinate system ($\hat{\mathbf{x}}_{1}$ corresponds to σ_{max} , $\hat{\mathbf{x}}_{3}$ corresponds to σ_{min}) for the stress tensors given.

	3	-1	0			3	0 (5]			10	-5	5
a.	- 1 0	3 0	0 1	,	b.	0 0	3 - -1	1 1	,	c.	- 5 5	0 5	5 10
	_					_					-]

Note: No numerical procedure is required to find the roots of the cubic equations.

SOLUTION: $(3-\lambda)(3-\lambda)(1-\lambda) - (1-\lambda) = (1-\lambda)[(3-\lambda)^2 - 1] = 0$ $\lambda = 1, \ \lambda = 2, \ \lambda = 4 \implies \sigma_1 = 4, \ \sigma_2 = 2, \ \sigma_3 = 1$ So. For $\sigma_1 = 4$, $\begin{vmatrix} -1 & -1 & 0 \\ -1 & -1 & 0 \\ 0 & 0 & -3 \end{vmatrix} \begin{vmatrix} n_1 \\ n_2 \\ n_3 \end{vmatrix} = \begin{vmatrix} 0 \\ 0 \\ 0 \end{vmatrix}$, where (n_1, n_2, n_3) are the components of \mathbf{x}_1 $n_3 = 0, n_1 + n_2 = 0, n_1^2 + n_2^2 = 1$ (unit vector) \therefore $\mathbf{n}_1 = \pm \frac{1}{\mathbf{\Omega}}, \ \mathbf{n}_2 = \pm \frac{1}{\mathbf{\Omega}}, \ \mathbf{n}_3 = 0, \ \text{or} \quad \mathbf{x}_1' \iff (\pm \frac{1}{\mathbf{\Omega}}, \pm \frac{1}{\mathbf{\Omega}}, 0)$ For $\sigma_2 = 2$ $\begin{vmatrix} 1 & -1 & 0 \\ -1 & 1 & 0 \\ 0 & 0 & -1 \end{vmatrix} \begin{vmatrix} m_1 \\ m_2 \\ m_3 \end{vmatrix} = \begin{vmatrix} 0 \\ 0 \\ 0 \end{vmatrix}$, where (m_1, m_2, m_3) are the components of \mathbf{x}_2 $m_3 = 0, m_1 - m_2 = 0, m_1^2 + m_2^2 = 1$ (unit vector) $\therefore \mathbf{m}_1 = \pm \frac{1}{\mathbf{\ell}^2}, \ \mathbf{m}_2 = \pm \frac{1}{\mathbf{\ell}^2}, \ \mathbf{m}_3 = 0, \ \mathrm{or} \quad \mathbf{x}_2' \iff (\pm \frac{1}{\mathbf{\ell}^2}, \pm \frac{1}{\mathbf{\ell}^2}, 0)$ For $\sigma_3 = 1$, $\begin{vmatrix} 2 & -1 & 0 \\ -1 & 2 & 0 \\ 0 & 0 & 0 \end{vmatrix} \begin{vmatrix} p_1 \\ p_2 \\ p_2 \end{vmatrix} = \begin{vmatrix} 0 \\ 0 \\ 0 \end{vmatrix}$, where (p_1, p_2, p_3) are the components of \mathbf{x}_3 $2p_1 - p_2 = 0$, $-p_1 + 2p_2 = 0 \implies p_1 = p_2 = 0$, $p_3^2 = \pm 1$ (unit vector) \therefore p₁ = 0, p₂ = 0, p₂ = ±1, or \mathbf{x}_{2} \leftrightarrow (0, 0, ±1)

In order to find the rotation matrix, we first choose a right-hand set from among the various choices of $\hat{\mathbf{x}}_1$, $\hat{\mathbf{x}}_2$, $\hat{\mathbf{x}}_3$

 $\hat{\mathbf{x}}_{1}' \leftrightarrow (\frac{1}{12}, -\frac{1}{12}, 0)$ $\hat{\mathbf{x}}_{2}' \leftrightarrow (\frac{1}{12}, \frac{1}{12}, 0)$ $\hat{\mathbf{x}}_{3}' \leftrightarrow (0, 0, 1) ,$ then the required rotation matrix is $\begin{bmatrix} \mathbf{R} \end{bmatrix} = \begin{bmatrix} \frac{1}{12} & -\frac{1}{12} & 0 \\ \frac{1}{12} & \frac{1}{12} & 0 \\ 0 & 0 & 1 \end{bmatrix}$

$$\begin{bmatrix} \sigma \end{bmatrix} = \begin{bmatrix} 3 & 0 & 0 \\ 0 & 3 & -1 \\ 0 & -1 & 1 \end{bmatrix}, \text{ so the eigen equation is } \begin{vmatrix} 3-\lambda & 0 & 0 \\ 0 & 3-\lambda & -1 \\ 0 & -1 & 1-\lambda \end{vmatrix} = 0$$

b.

c.

$$(3 - \lambda) [(3 - \lambda) (1 - \lambda) - 1] = 0$$

So, $\lambda_1 = 3, \ \lambda_{2,3} = \frac{4 \pm \sqrt{8}}{2} = 3.41, \ 0.59 \implies \sigma_1 = 3.414, \ \sigma_2 = 3.000, \ \sigma_3 = 0.586$

For
$$\sigma_1 = 3.41$$
, $\begin{bmatrix} -0.41 & 0 & 0 \\ 0 & -0.41 & -1 \\ 0 & -1 & -2.41 \end{bmatrix} \begin{bmatrix} n_1 \\ n_2 \\ n_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$, where (n_1, n_2, n_3) are the components of \mathbf{x}_1 , $\mathbf{x}_1 = 0, -0.41 n_2 - n_3 = 0, n_1^2 + n_2^2 + n_3^2 = 1$
 $\therefore \mathbf{x}_1 \leftrightarrow (0, \pm 0.92, \pm 0.38)$
For $\sigma_2 = 3$, $\begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & -1 \\ 0 & -1 & -2 \end{bmatrix} \begin{bmatrix} m_1 \\ m_2 \\ m_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$, where (m_1, m_2, m_3) are the components of \mathbf{x}_2 , $\mathbf{x}_3 = 0, m_2 = 0, m_1 = \pm 1$
 $\therefore \mathbf{x}_2 \leftrightarrow (\pm 1, 0, 0)$
For $\sigma_3 = 0.586$, $\begin{bmatrix} 2.414 & 0 & 0 \\ 0 & 2.414 & -1 \\ 0 & -1 & 0.414 \end{bmatrix} \begin{bmatrix} p_1 \\ p_2 \\ p_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$, where (p_1, p_2, p_3) are the components of \mathbf{x}_3 , $\mathbf{x}_3 = \mathbf{x}_3 = 0, \mathbf{x}_$

In order for find the rotation matrix we first choose a right-handed set of eigenvectors: $\mathbf{\hat{x}}_{1}^{'} \leftrightarrow (0, 0.92, -0.38)$ $\mathbf{\hat{x}}_{2}^{'} \leftrightarrow (1, 0, 0)$ $\mathbf{\hat{x}}_{3}^{'} \leftrightarrow (0, -0.38, -0.92)$ then the rotation matrix is $\begin{bmatrix} R \end{bmatrix} = \begin{bmatrix} 0 & 0.92 & -0.38 \\ 1 & 0 & 0 \\ 0 & -0.38 & -0.92 \end{bmatrix}$

$$\begin{bmatrix} \sigma \end{bmatrix} = \begin{bmatrix} 10 - 5 & 5 \\ -5 & 0 & 5 \\ 5 & 5 & 10 \end{bmatrix} \Rightarrow \sigma_1 = 15, \ \sigma_2 = 10, \ \sigma_3 = -5$$

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$$\hat{\mathbf{x}}_{1}^{'} \leftrightarrow (0.707, 0, 0.707), \text{ for } \boldsymbol{\sigma}_{1} = 15$$

$$\hat{\mathbf{x}}_{2}^{'} \leftrightarrow (0.577, -0.577, -0.577), \text{ for } \boldsymbol{\sigma}_{2} = 10$$

$$\hat{\mathbf{x}}_{3}^{'} \leftrightarrow (0.408, 0.816, -.408), \text{ for } \boldsymbol{\sigma}_{3} = -5$$

$$\begin{bmatrix} R \end{bmatrix} = \begin{bmatrix} .707 & 0 & .707 \\ .577 & -.577 & .577 \\ .408 & .816 & -.408 \end{bmatrix}$$

$$\begin{bmatrix} R \end{bmatrix} \begin{bmatrix} \sigma \end{bmatrix} \begin{bmatrix} R \end{bmatrix}^{T} = \begin{bmatrix} \sigma_{1} & 0 & 0 \\ 0 & \sigma_{2} & 0 \\ 0 & 0 & \sigma_{3} \end{bmatrix}$$

$$\begin{bmatrix} R \end{bmatrix} \begin{bmatrix} \sigma \end{bmatrix} \begin{bmatrix} \sigma \end{bmatrix} \begin{bmatrix} 10.6 & 0 & 10.6 \\ 5.77 & -5.77 & -5.77 \\ -2.04 & -4.08 & 2.04 \end{bmatrix}$$

$$\begin{bmatrix} R \end{bmatrix} \begin{bmatrix} \sigma \end{bmatrix} \begin{bmatrix} R \end{bmatrix}^{T} = \begin{bmatrix} 15 & 0 & 0 \\ 0 & 10 & 0 \\ 0 & 0 & -5 \end{bmatrix}$$

Check:

4. Find the invariants for the stress tensors shown below:

	1.44	0.22	-0.76		1.75	0.35	-0.75		- 1.94	0.38	-0.54
a.	0.22	2.25	-0.38	, b.	0.35	2.50	-0.35	, c.	0.38	2.75	- 0.22
	-0.76	-0.38	2.31		-0.75	-0.35	1.75		-0.54	-0.22	1.31

SOLUTION:

$$\begin{bmatrix} \sigma \end{bmatrix} = \begin{bmatrix} 1.44 & 0.22 & -0.76 \\ 0.22 & 2.25 & -0.38 \\ -0.76 & -0.38 & 2.31 \end{bmatrix}$$

a.

$$J_{1} = \sigma_{11} + \sigma_{22} + \sigma_{33} = 1.44 + 2.25 + 2.31 = 6$$

$$J_{2} = -\left(\sigma_{11} \sigma_{22} + \sigma_{22} \sigma_{33} + \sigma_{33} \sigma_{11}\right) + \sigma_{23}^{2} + \sigma_{31}^{2} + \sigma_{12}^{2}$$

$$= -\left[\left(1.44\right)\left(2.25\right) + \left(2.25\right)\left(2.31\right) + \left(2.31\right)\left(1.44\right)\right] + \left(-0.38\right)^{2} + \left(-0.76\right)^{2} + \left(0.22\right)^{2} = -11$$

$$J_{3} = \begin{vmatrix} 1.44 & 0.22 & -0.76 \\ 0.22 & 2.25 & -0.38 \\ -0.76 - 0.38 & 2.31 \end{vmatrix} = 6$$

$$[\sigma] = \begin{bmatrix} 1.75 & 0.35 & -0.75 \\ 0.35 & 2.50 & -0.35 \\ -0.75 & -0.35 & 1.75 \end{vmatrix}$$

$$J_{1} = 6$$

b.

$$J_{1} = 6$$

 $J_{2} = -11$
 $J_{3} = 6$

$$[\sigma] = \begin{bmatrix} 1.94 & 0.38 & -0.54 \\ 0.38 & 2.75 & -0.22 \\ -0.54 & -0.22 & 1.31 \end{bmatrix}$$
c.

$$J_1 = 6$$

 $J_2 = -11$
 $J_3 = 6$

5. Find and solve the characteristic equations for the stress tensors shown in Problem 4. Use the method followed in Exercise 3.3. (A numerical procedure is required.)

SOLUTION:

- $J_1 = 6$
- $J_2 = -11$ $J_3 = 6$
- a.

 $\begin{aligned} 0 &= \lambda^3 - 6\lambda^2 + 11\lambda - 6 = \varphi(\lambda) \\ i \quad \lambda_i \quad \varphi(\lambda_i) \quad \varphi'(\lambda_i) \quad \Delta\lambda \\ 0 \quad 1 \quad 0 \qquad \left[\lambda = 1 \text{ exactly. Lucky guess!}\right] \end{aligned}$

To obtain the quadratic equation, perform synthetic long division as shown below.

$$\begin{array}{r} \lambda^2 & -5\lambda & +6 \\ (\lambda - 1)\overline{\lambda^3 - 6\lambda^2 + 11\lambda - 6} \\ \lambda^3 - 6\lambda^2 \\ & -5\lambda^2 + 11\lambda \\ & \underline{-5\lambda^2 + 5\lambda} \\ & 6\lambda - 6 \\ & 6\lambda - 6 \end{array}$$
thus the original expression is $(\lambda - 1) (\lambda^2 - 5\lambda + 6)$

The remaining roots are found by the quadratic formula:

$$\lambda = \frac{+5 \pm \sqrt{25 - 24}}{2} = 2, \ 3$$

So, the three roots (principal stresses) are:

 $\sigma_1 = 3, \quad \sigma_2 = 2, \quad \sigma_3 = 1$

and the characteristic equation can be written in product form: $(\lambda - 3)(\lambda - 2)(\lambda - 1) = 0$

b.
$$J_1 = 6, J_2 = -11, J_3 = 6$$

 $0 = \lambda^3 - 6\lambda^2 + 11\lambda - 6 = \varphi(\lambda)$

This characteristic equation is identical to 5.a., thus the two stress tensors are identical except for a rotation. The principal values must therefore be the same.

c.
$$J_1 = 6, J_2 = -11, J_3 = 6$$

[Same as 5.a. and 5.b.]

6. Find the principal directions for the stress tensors shown in Problem 4 and find the rotation matrix which transforms components given in the original coordinate system to ones in a principal coordinate system. (Assume that the minimum principal stress acts on a plane with normal $\hat{\mathbf{x}}_1$ ' and the maximum principal stress acts on a plane with normal $\hat{\mathbf{x}}_3$ '.)

SOLUTION:

$$\sigma_{1} = 1 \qquad \qquad \mathbf{\hat{x}}_{1}^{'} \leftrightarrow \pm (0.866, 0, 0.5)$$

$$\sigma_{2} = 2 \qquad \Rightarrow \qquad \mathbf{\hat{x}}_{2}^{'} \leftrightarrow \pm (-0.24, 0.866, 0.433)$$

$$\sigma_{3} = 3 \qquad \qquad \mathbf{\hat{x}}_{3}^{'} \leftrightarrow \pm (-0.43, -0.5, 0.75)$$

a.

b.

c.

Taking the three plus signs forms a right-handed system for which the rotation matrix is

$$\begin{bmatrix} \mathbf{R} \end{bmatrix} = \begin{bmatrix} 0.866 & 0 & 0.5 \\ -0.24 & 0.866 & 0.433 \\ -0.43 & -0.5 & 0.75 \end{bmatrix}.$$

$$\boldsymbol{\sigma}_{1} = 1 \qquad \qquad \mathbf{\hat{x}}_{1} = \pm (-0.707, \ 0, \ -0.707) \\ \boldsymbol{\sigma}_{2} = 2 \implies \mathbf{\hat{x}}_{2} = \pm (-0.5, \ 0.704, \ 0.50) \\ \boldsymbol{\sigma}_{3} = 3 \qquad \qquad \mathbf{\hat{x}}_{3} = \pm (0.50, \ 0.704, \ -0.50) \end{bmatrix}.$$

For the "+" signs (one choice of right-handed system):

$$\begin{bmatrix} \mathbf{R} \end{bmatrix} = \begin{bmatrix} -0.707 & 0 & -0.707 \\ -0.50 & 0.704 & 0.50 \\ 0.50 & 0.704 & -0.50 \end{bmatrix}$$
$$\boldsymbol{\sigma}_{1} = 1 \qquad \qquad \mathbf{\hat{x}}_{1} = \pm (0.5, \ 0, \ 0.867)$$
$$\boldsymbol{\sigma}_{2} = 2 \qquad \Rightarrow \qquad \mathbf{\hat{x}}_{2} = \pm (-0.75, \ 0.5, \ 0.43)$$
$$\boldsymbol{\sigma}_{3} = 3 \qquad \qquad \mathbf{\hat{x}}_{3} = \pm (-0.433, \ -0.867, \ 0.24)$$

For the "+" signs (one choice of right-handed system):

$$\begin{bmatrix} \mathbf{R} \end{bmatrix} = \begin{bmatrix} 0.5 & 0 & 0.867 \\ -0.75 & 0.5 & 0.43 \\ -0.43 & -0.867 & 0.24 \end{bmatrix}$$

7. Find the spherical and deviatoric components of the stress tensors given in Problem 4. Find the principal stresses and directions of the deviatoric tensors following the method outlined in Section 3.6. How do these compare with the values for the stress tensor obtained in Problems 5 and 6.

SOLUTION:

a.

stress	S	pherica	al	deviatoric			
$\begin{bmatrix} 1.44 & 0.22 & -0.76 \\ 0.22 & 2.25 & -0.38 \\ -0.76 & -0.38 & 2.31 \end{bmatrix}$	=	$\begin{bmatrix} 2 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 2 \end{bmatrix}$	+	-0.56 0.22 -0.76	0.22 - 0.25 - -0.38	-0.76 -0.38 0.31	

$$\alpha_{1} = \frac{1}{3} \cos^{-1}(0) = 30^{\circ}$$

$$\alpha_{2} = 120^{\circ} + 30^{\circ} = 150^{\circ}$$

$$\alpha_{3} = 30^{\circ} - 120^{\circ} = -90^{\circ}$$

Spherical Invariants: $J_2^d \approx 1.0 \quad J_3^d \approx 0$, so

$$\sigma_{1}^{d} = 2\left(\frac{1}{3}\right)^{\frac{1}{2}}\cos(30^{\circ}) = 1.0$$

$$\sigma_{2}^{d} = 2\left(\frac{1}{3}\right)^{\frac{1}{2}}\cos(150^{\circ}) = -1.0$$

$$\sigma_{3}^{d} = 2\left(\frac{1}{3}\right)^{\frac{1}{2}}\cos(-90^{\circ}) = 0.0$$

Therefore:

$$\sigma_{1} = 1.0 + 2.0 = 3$$

$$\sigma_{2} = -1.0 + 2.0 = 1$$

$$\sigma_{3} = 0 + 2.0 = 2$$

The principal directions are found using the principal values with the same result as Problems 5.a. and 6.a.

stress spherical deviatoric

$$\begin{bmatrix} 1.75 & 0.35 & -0.75\\ 0.35 & 2.50 & -0.35\\ -0.75 & -0.35 & 1.75 \end{bmatrix} = \begin{bmatrix} 2 & 0 & 0\\ 0 & 2 & 0\\ 0 & 0 & 2 \end{bmatrix} + \begin{bmatrix} -0.56 & 0.22 & -0.76\\ 0.22 & 0.25 & -0.38\\ -0.76 & -0.38 & 0.31 \end{bmatrix}$$

b.

The results are the same as Problems 5.b. and 6.b.

stresssphericaldeviatoric
$$\begin{bmatrix} 1.94 & 0.38 & -0.54 \\ 0.38 & 2.75 & -0.22 \\ -0.54 & -0.22 & 1.31 \end{bmatrix} = \begin{bmatrix} 2 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 2 \end{bmatrix} + \begin{bmatrix} -0.56 & 0.22 & -0.76 \\ 0.22 & 0.25 & -0.38 \\ -0.76 & -0.38 & 0.31 \end{bmatrix}$$

c.

The results are the same as Problems 5.c. and 6.c.

- 8. Find the spherical, deviatoric, principal deviatoric components, and principal directions of stress for the following cases:
 - Uniaxial tension: $\sigma_{11} = \sigma$, other $\sigma_{ij} = 0$ Simple shear: $\sigma_{21} = \sigma_{12} = \sigma$, other $\sigma_{ij} = 0$ Balanced biaxial tension: $\sigma_{11} = \sigma_{22} = \sigma$, other $\sigma_{ij} = 0$ Biaxial shear: $\sigma_{13} = \sigma_{31} = \sigma_A, \sigma_{21} = \sigma_{12} = \sigma_B$, other $\sigma_{ij} = 0$ Tension and shear: $\sigma_{11} = \sigma_t, \sigma_{13} = \sigma_{31} = \sigma_s$, other $\sigma_{ij} = 0$

SOLUTION:

stress spherical deviatoric $\begin{bmatrix} 1 \\ 0 \end{bmatrix} \begin{bmatrix} 1 \\ 0 \end{bmatrix} \begin{bmatrix} 2 \\ 0 \end{bmatrix}$

	$\sigma 0 0$		$\frac{\sigma}{3} 0 0$		$\frac{2\sigma}{3}$ 0 0
	000	=	$0\frac{\sigma}{3}0$	+	$0 -\frac{\sigma}{3} 0$
a. Uniaxial tension $\left(\sigma_{11} = \sigma, \text{ other } \sigma_{ij} = 0\right)$,	0 0 0		$0 0 \frac{\sigma}{3}$		$0 0 -\frac{\sigma}{3}$

$$\sigma_{1}^{d} = \frac{2\sigma}{3} \qquad \sigma_{2}^{d} = \sigma_{3}^{d} = -\frac{\sigma}{3}$$

$$\widehat{\mathbf{m}} \leftrightarrow (1, 0, 0)$$

$$\widehat{\mathbf{n}} \leftrightarrow (0, 1, 0) \qquad \text{(current axes are principal)}$$

$$\widehat{\mathbf{p}} \leftrightarrow (0, 0, 1)$$

b. Simple shear
$$\left(\sigma_{12} = \sigma, \text{ other } \sigma_{ij} = 0\right)$$
, stress spherical deviatoric $\begin{bmatrix} 0 & \sigma & 0 \\ \sigma & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} + \begin{bmatrix} 0 & \sigma & 0 \\ \sigma & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$

$$\begin{split} & \widehat{\mathbf{m}} \leftrightarrow \begin{pmatrix} \frac{1}{\mathbf{1}^2}, \frac{1}{\mathbf{1}^2}, 0 \end{pmatrix} \\ & \widehat{\mathbf{n}} \leftrightarrow \begin{pmatrix} 0, 0, 1 \end{pmatrix} \\ & \widehat{\mathbf{p}} \leftrightarrow \begin{pmatrix} \frac{1}{\mathbf{1}^2}, -\frac{1}{\mathbf{1}^2}, 0 \end{pmatrix} \end{split}$$

stress spherical deviatoric

$$\begin{bmatrix} \sigma & 0 & 0 \\ 0 & \sigma & 0 \\ 0 & \sigma & 0 \\ 0 & 0 & 0 \end{bmatrix} = \begin{bmatrix} \frac{2\sigma}{3} & 0 & 0 \\ 0 & \frac{2\sigma}{3} & 0 \\ 0 & 0 & \frac{2\sigma}{3} \end{bmatrix} + \begin{bmatrix} \frac{\sigma}{3} & 0 & 0 \\ 0 & \frac{\sigma}{3} & 0 \\ 0 & 0 & \frac{2\sigma}{3} \end{bmatrix}$$
c. Balanced biaxial tension ($\sigma_{11} = \sigma_{22} = 0$, other $\sigma_{ij} = 0$)

$$\sigma_{1}^{d} = \sigma_{2}^{d} = \frac{\sigma}{3} \qquad \sigma_{3}^{d} = -\frac{2\sigma}{3} \qquad \hat{\mathbf{m}} \leftrightarrow (1, 0, 0)$$
$$\hat{\mathbf{n}} \leftrightarrow (0, 1, 0)$$
$$\hat{\mathbf{p}} \leftrightarrow (0, 0, 1)$$

(current axes are principal)

d. Biaxial shear ($\sigma_{13} = \sigma_{31} = \sigma_A$, $\sigma_{12} = \sigma_{21} = \sigma_B$, other $\sigma_{ij} = 0$)

e. Tension and shear: $\sigma_{11} = \sigma_t$, $\sigma_{13} = \sigma_{31} = \sigma_s$, other $\sigma_{ij} = 0$

stress spherical deviatoric

$$\begin{bmatrix} \sigma_{t} & 0 & \sigma_{3} \\ 0 & 0 & 0 \\ \sigma_{s} & 0 & 0 \end{bmatrix} = \begin{bmatrix} \frac{\sigma_{t}}{3} & 0 & 0 \\ 0 & \frac{\sigma_{t}}{3} & 0 \\ 0 & 0 & \frac{\sigma_{t}}{3} \end{bmatrix} + \begin{bmatrix} \frac{2\sigma_{t}}{3} & 0 & \sigma_{s} \\ 0 & -\frac{\sigma_{t}}{3} & 0 \\ \sigma_{s} & 0 & -\frac{\sigma_{t}}{3} \end{bmatrix}$$

$$\sigma_{1} = \frac{\sigma_{t} + \sqrt{\sigma_{t}^{2} + 4\sigma_{s}^{2}}}{2}$$

$$\sigma_{2} = 0$$

$$\sigma_{3} = \frac{\sigma_{t} - \sqrt{\sigma_{t}^{2} + \sigma_{s}^{2}}}{2}$$

$$\widehat{\mathbf{m}} \leftrightarrow \left(\frac{\sigma_{s}}{D}, 0, \frac{\sigma_{t} - \sigma_{1}}{D}\right), \text{ where } D = \left[\left(\sigma_{t} - \sigma_{1}\right)^{2} + \sigma_{s}^{2}\right]^{\frac{1}{2}}$$

$$\widehat{\mathbf{n}} \leftrightarrow \left(0, 1, 0\right)$$

$$\widehat{\mathbf{p}} \leftrightarrow \pm \left(\frac{\sigma_{s}}{D'}, 0, \frac{\sigma_{t} - \sigma_{2}}{D'}\right), \text{ where } D = \left[\left(\sigma_{t} - \sigma_{2}\right)^{2} + \sigma_{s}^{2}\right]^{\frac{1}{2}}$$

(If $\sigma_s \sigma_t < 0$, the minus sign is adopted for the components of **p**.)

B. DEPTH PROBLEMS

9. The reciprocal theorem of Cauchy states that the stress vectors acting on two intersecting planes have the following property:

$$\mathbf{s}_1 \cdot \, \mathbf{\hat{n}}_2 = \, \mathbf{s}_2 \cdot \, \mathbf{\hat{n}}_1$$

where s_i is the stress vector acting on a plane with normal $\hat{\mathbf{n}}_i$. Show that this principle follows from the symmetry of the stress tensor, or from the equilibrium condition directly.

SOLUTION:

It is possible to prove the relationship by considering Cauchy's tetrahedron (Exercise 3.1), or by multiplying all of the required components and comparing the results.

The shortest method is by writing the various terms in indicial notation.

Let
$$\widehat{\mathbf{n}}_1 = \widehat{\mathbf{n}}, \ \widehat{\mathbf{n}}_2 = \widehat{\mathbf{m}}, \text{ and } \mathbf{s}_1 = \mathbf{s} \text{ and } \mathbf{s}_2 = \mathbf{t}$$
 for simpler notation, then
 $\mathbf{s} = \sigma \widehat{\mathbf{n}} \iff \mathbf{s}_i = \sigma_{ij} \mathbf{n}_j$
 $\mathbf{t} = \sigma \widehat{\mathbf{m}} \iff \mathbf{t}_i = \sigma_{ij} \mathbf{m}_j$
 $\mathbf{s} \cdot \widehat{\mathbf{m}} = \sigma \widehat{\mathbf{n}} \widehat{\mathbf{m}} \text{ or, } \mathbf{s} \cdot \widehat{\mathbf{m}} = \mathbf{s}_i \mathbf{m}_i = \sigma_{ij} \mathbf{n}_j \mathbf{m}_i$
 $\mathbf{t} \cdot \widehat{\mathbf{n}} = \sigma \widehat{\mathbf{m}} \widehat{\mathbf{n}} \text{ or } \mathbf{t} \cdot \widehat{\mathbf{n}} = \mathbf{t}_i \mathbf{n}_i = \sigma_{ij} \mathbf{m}_j \mathbf{n}_i$
but, since $\sigma_{ij} = \sigma_{jj}$ we can rewrite $\sigma_{ij} \mathbf{n}_j \mathbf{m}_i = \sigma_{ij} \mathbf{n}_i \mathbf{m}_j$, so $\mathbf{s} \cdot \widehat{\mathbf{m}} = \mathbf{t} \cdot \widehat{\mathbf{n}}$.

10. Octahedral planes are ones which have normals forming equal angles with the three principal axes. Find an expression for S_n , the normal components of the stress vector on the octahedral plane in terms of a) principal stresses and b) arbitrary stress components.

SOLUTION:



$$\begin{bmatrix} \boldsymbol{\sigma} \end{bmatrix} = \begin{bmatrix} \boldsymbol{\sigma}_1 & \boldsymbol{0} & \boldsymbol{0} \\ \boldsymbol{0} & \boldsymbol{\sigma}_2 & \boldsymbol{0} \\ \boldsymbol{0} & \boldsymbol{0} & \boldsymbol{\sigma}_3 \end{bmatrix}, \text{ and } \begin{bmatrix} \boldsymbol{n} \end{bmatrix} = \begin{bmatrix} \frac{1}{73} \\ \frac{1}{73} \\ \frac{1}{73} \\ \frac{1}{73} \end{bmatrix}$$

a. In principal axes, as show in the figure above,

$$\mathbf{S}_{n} = \mathbf{n} \cdot \mathbf{S} \cdot \mathbf{n} = \begin{bmatrix} \frac{1}{\sqrt{3}} & \frac{1}{\sqrt{3}} & \frac{1}{\sqrt{3}} \end{bmatrix} \begin{bmatrix} \sigma_{1} & 0 & 0 \\ 0 & \sigma_{2} & 0 \\ 0 & 0 & \sigma_{3} \end{bmatrix} \begin{bmatrix} \frac{1}{\sqrt{3}} \\ \frac{1}{\sqrt{3}} \\ \frac{1}{\sqrt{3}} \end{bmatrix} = \frac{1}{3} (\sigma_{1} + \sigma_{2} + \sigma_{3}) \mathbf{p} = \frac{1}{3} \mathbf{J}_{1} (\text{Eq. 2.38})$$

Therefore:

where J_1 is the first stress invariant.

b. Since we know that this quantity is invariant to the choice of coordinate system orientation, the expression in an arbitrary cartesian system is

 $S_n = p = \frac{1}{3}J_1 = \frac{1}{3}(\sigma_{11} + \sigma_{22} + \sigma_{33})$

11. Show that the tangential component of the stress vector on the octahedral planes (i.e. the shear $\left(\frac{2}{3}J_2'\right)^{\frac{1}{2}}$, where J_2' is the second invariant of the deviatoric stress tensor.

SOLUTION:

As shown in Problem 10., the normal component on an octahedral plane is

$$\mathbf{S}_{\mathrm{N}} = \frac{1}{3} \left(\boldsymbol{\sigma}_1 + \boldsymbol{\sigma}_2 + \boldsymbol{\sigma}_3 \right)$$

The tangential component may be found from the relationship

$$\mathbf{S}_{N}^{2} + \mathbf{S}_{T}^{2} = \mathbf{S} \cdot \mathbf{S} = (\mathbf{\sigma} \cdot \mathbf{n}) \cdot (\mathbf{\sigma} \cdot \mathbf{n}), \text{ or}$$

$$\mathbf{S}_{T}^{2} = (\mathbf{\sigma} \cdot \mathbf{n}) \cdot (\mathbf{\sigma} \cdot \mathbf{n}) - \mathbf{S}_{N}^{2} = \frac{1}{3} \left(\sigma_{1}^{2} + \sigma_{2}^{2} + \sigma_{3}^{2} \right) - \frac{1}{9} \left(\sigma_{1} + \sigma_{2} + \sigma_{3} \right)^{2}$$

$$9 \mathbf{S}_{T}^{2} = 2 \sigma_{1}^{2} + 2 \sigma_{2}^{2} + 2 \sigma_{3}^{2} - 2 \sigma_{1} \sigma_{2} - 2 \sigma_{1} \sigma_{3} - 2 \sigma_{2} \sigma_{3} = 6 \mathbf{J}_{2}$$

So,
$$\mathbf{S}_{\mathrm{T}} = \left(\frac{2}{3} \mathbf{J}_{2}\right)^{\frac{1}{2}}$$

12. Problems involving cylindrical symmetry often use cylindrical coordinates r, θ , z where $r = \left(x_1^2 + x_2^2\right)^{\frac{1}{2}}$ $\theta = \tan^{-1} \frac{x}{y}$. (Conversely, $x_1 = r \cos \theta$, $x_2 = r \sin \theta$, $x_3 = z$). For example, consider pure torsion of an elastic, long bar with axis parallel to \hat{z} , where $\sigma_{\theta z}$ is a constant on the

outside of the bar. Find the stress tensor in two alternate Cartesian coordinate systems:

- a. One which has $\hat{\mathbf{x}}_1$ normal to the cylinder axis and is tangent to the cylinder surface, $\hat{\mathbf{x}}_2$ normal to the cylinder surface, and $\hat{\mathbf{x}}_3$ the cylinder axis.
- b. One which is fixed in space (i.e. independent of θ), with $\hat{\mathbf{x}}_3 //\hat{\mathbf{z}}$, $\hat{\mathbf{x}}_1$ lying in the $\theta=0$, z=0 direction, and $\hat{\mathbf{x}}_2$ lying in the $\theta=2$, z=0 direction.



Since the axes are identical, the stress components are identical, i.e.



b.

(In this case, $\hat{\mathbf{r}}$ and $\hat{\boldsymbol{\theta}}$ rotate as the point of interest rotates, but hte cartesian system is fixed.) $\begin{bmatrix} \sigma' \end{bmatrix} = \begin{bmatrix} \mathbf{R} \end{bmatrix} \begin{bmatrix} \sigma \end{bmatrix} \begin{bmatrix} \mathbf{R} \end{bmatrix}^{\mathrm{T}} = \begin{bmatrix} \cos\theta & -\sin\theta & 0\\ \sin\theta & \cos\theta & 0\\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 0 & 0 & 0\\ 0 & \sigma & 0\\ 0 & \sigma & 0 \end{bmatrix} \begin{bmatrix} \cos\theta & \sin\theta & 0\\ -\sin\theta & \cos\theta & 0\\ 0 & 0 & 1 \end{bmatrix}$

a.

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$$= \begin{bmatrix} \cos \theta - \sin \theta & 0\\ \sin \theta & \cos \theta & 0\\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 0 & 0 & 0\\ 0 & 0 & \sigma\\ \sigma(-\sin \theta) & \sigma \cos \theta & 0 \end{bmatrix}$$
$$[\sigma'] = \begin{bmatrix} 0 & 0 & -\sigma \sin \theta\\ 0 & \sigma & \sigma \cos \theta\\ -\sigma & \sin \theta & \sigma & \cos \theta & 0 \end{bmatrix}$$

(fixed Cartesian system)

13. Show that if two roots of the characteristic equation are identical (i.e. degenerate), then any direction normal to the other principal direction (i.e. the one corresponding to the non-identical root) is a principal direction. Show that if all three roots are identical, all directions are principal.

SOLUTION:

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a. Assume that the characteristic equation is of the form:

 $\left(\!\lambda-\sigma_{\scriptscriptstyle 0}\!\right)\left(\!\lambda-\sigma_{\scriptscriptstyle 1}\!\right)^{\scriptscriptstyle 2} = 0 \ , \label{eq:constraint}$

0

where σ_1 is a degenerate root. The stress components in the principal axes are

 $\begin{bmatrix} \sigma_{_1} & 0 & 0 \\ 0 & \sigma_{_1} & 0 \\ 0 & 0 & \sigma_{_o} \end{bmatrix}$

where σ_{0} is the principal stress in the x_{3} (3rd principal) direction. A general rotation of coordinate system about the x_{3} axis may be written as follows:

$\begin{bmatrix} \sigma \end{bmatrix} = \begin{bmatrix} \cos \theta & \sin \theta \\ -\sin \theta & \cos \theta \\ 0 & 0 \end{bmatrix}$	$\begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} \begin{bmatrix} \sigma_1 \\ 0 \\ 0 \end{bmatrix}$	$\begin{bmatrix} 0 & 0 \\ \sigma_1 & 0 \\ 0 & \sigma_0 \end{bmatrix} \begin{bmatrix} \cos \theta \\ \sin \theta \\ 0 \end{bmatrix}$	$\theta - \sin \theta$ $\cos \theta$ 0	0 0 1	=
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$$\sigma_1 \left(\cos^2 \theta + \sin^2 \theta \right) \qquad \sigma_1 \left(-\sin \theta \cos \theta + \sin \theta \cos \theta \right)$$

$$= \left[\sigma_1 \left(\sin \theta \cos \theta - \sin \theta \cos \theta \right) \qquad \sigma_1 \left(\sin^2 \theta + \cos^2 \theta \right) \qquad 0$$

0

0

 σ_{0}

$$\boldsymbol{\sigma} = \begin{bmatrix} \boldsymbol{\sigma}_1 & \boldsymbol{0} & \boldsymbol{0} \\ \boldsymbol{0} & \boldsymbol{\sigma}_1 & \boldsymbol{0} \\ \boldsymbol{0} & \boldsymbol{0} & \boldsymbol{\sigma}_0 \end{bmatrix} \qquad \left(\text{for any direction normal to } \hat{\boldsymbol{x}}_3 \right)$$

- b. If all three roots are degenerate, there are many ways to show that any direction is equivalent. The derivation in Part a. can be done for $\sigma_1 = \sigma_0$, or one can note that all three roots being equivalent is the same as the spherical component (i.e. hydrostatic pressure or tension).
- 14. It is often convenient to replace one set of forces with another, statically-equivalent set. For example, consider a triangular element of material (assume unit depth normal to the triangle) which is assumed to be a small enough piece of a body to feel only a homogeneous stress, σ_{ij} (i,j=1,2, assuming that $\sigma_{i3}=0$, where $\hat{\mathbf{x}}_3$ is normal to the triangle). Use a simple, physically-motivated procedure to replace σ_{ij} by three forces, f_1 , f_2 , f_3 acting at the three corners of the triangle. Consider the force transmitted by each face.

Consider a triangle with normals defined to each side with a magnitude equal to the length of the side. (For unit depth of the sides in three dimensions, these are area vectors corresponding to the sides.)



A, **B**, **C** are deduced from **a**, **b**, and **c** by a rotation of -90°, therefore since $\mathbf{a} + \mathbf{b} + \mathbf{c} = 0$, $\mathbf{A} + \mathbf{B} + \mathbf{C} = 0$. The forces acting on the planes **A**, **B**, and **C** are $\mathbf{f}_{A} = \boldsymbol{\sigma} \mathbf{A}$, $\mathbf{f}_{B} = \boldsymbol{\sigma} \mathbf{B}$, $\mathbf{f}_{C} = \boldsymbol{\sigma} \mathbf{C}$ and $\mathbf{f}_{A} + \mathbf{f}_{B} + \mathbf{f}_{C} = \boldsymbol{\sigma} (\mathbf{A} + \mathbf{B} + \mathbf{C}) = 0$ because $\mathbf{A} + \mathbf{B} + \mathbf{C} = 0$.

To assign these forces to the vertices, let's use the unweighted average (although other choices might make more physical sense) of the forces on the connected sides:

$$\mathbf{f}_{1} = \frac{1}{2} \left(\mathbf{f}_{A} + \mathbf{f}_{C} \right) = \frac{1}{2} \boldsymbol{\sigma} \left(\mathbf{A} + \mathbf{C} \right)$$
$$\mathbf{f}_{2} = \frac{1}{2} \left(\mathbf{f}_{A} + \mathbf{f}_{B} \right) = \frac{1}{2} \boldsymbol{\sigma} \left(\mathbf{A} + \mathbf{B} \right)$$
$$\mathbf{f}_{3} = \frac{1}{2} \left(\mathbf{f}_{B} + \mathbf{f}_{C} \right) = \frac{1}{2} \boldsymbol{\sigma} \left(\mathbf{B} + \mathbf{C} \right)$$

15. Physically, why can the entire material loading at a point be reduced to three orthogonal force intensities passing through the point? Why do the shear components disappear along these directions?

SOLUTION:

For simplicity, let's consider a two-dimensional situation first. Similar to Figure 3.1, imagine making an arbitrary mathematical cut as shown in part (a) of the figure below. We can find the force acting on one of the cut faces required to maintain equilibrium, part (b). (The opposite force is required on the other cut face by equilibrium.) Then, glue the first cut back togethher and using the direction of the force as a guide, make another cut, this one perpendicular to the force observed on the first cut. Find the new force required for equilibrium and, if necessary, make another cut perpendicular to the new force. Continue until the current cut and current force are perpendicular, part (c). If we now relax the forces on the cut plane (and any external forces required to maintain equilibrium as the cut face is unloaded), can we be assured that the material is completely unloaded? No, because the direction parallel to the cut face (grey arrow in part (c)) is unaffected by the cut and therefore we have no information about it. Therefore, make a cut perpendicular to the first force, part (d). This simple thought exercise demonstrates why there are only two independent force intensities passing through a point in a two-dimensional body, and why they must be perpendicular.

To extend the exercise to three dimensions, follow precisely the same procedure. Once the first plane and normal force are found, there remain two perpendicular planes which must have only normal forces acting on them.



Although opposite to the usual derivation, it would be possible to derive the symmetry of the stress tensor by first noting that this result requires the existence the three perpendicular principal directions and that any rotation of axes from this principal set must produce a symmetric and real set of tensor components.

16. The two sets of components presented below correspond to the identical stress tensor, as measured in two coordinate systems, $\hat{\mathbf{x}}_1$, $\hat{\mathbf{x}}_2$, $\hat{\mathbf{x}}_3$, and $\hat{\mathbf{x}}_1$, $\hat{\mathbf{x}}_2$, $\hat{\mathbf{x}}_3$. Find the rotation matrix to transform components from the $\hat{\mathbf{x}}_i$ system to the $\hat{\mathbf{x}}_i$ ' system, and vice versa. (Hint: First find the rotations to the common, principal coordinate systems.)

$$\begin{bmatrix} \sigma \end{bmatrix} = \begin{bmatrix} 1.000 & 1.730 & 1.000 \\ 1.730 & 0.750 & 0.433 \\ 1.000 & 0.433 & 0.250 \end{bmatrix} \quad \begin{bmatrix} \sigma' \end{bmatrix} = \begin{bmatrix} 0.500 & 1.414 & 0.500 \\ 1.414 & 1.000 & 1.414 \\ 0.500 & 1.414 & 0.500 \end{bmatrix}$$

SOLUTION: $[\sigma'] = [R] [\sigma] [R]^{T}$

Define rotation matrices $[R_1]$ and $[R_2]$ such that $[\sigma_{\text{principal}}] = [R_1] [\sigma] [R_1]^T$

$$\begin{bmatrix} \boldsymbol{\sigma}_{\text{principal}} \end{bmatrix} = \begin{bmatrix} \mathbf{R}_2 \end{bmatrix} \begin{bmatrix} \boldsymbol{\sigma}' \end{bmatrix} \begin{bmatrix} \mathbf{R}_2 \end{bmatrix}^{\mathrm{T}}$$

Then, we can find [R] in terms of $\begin{bmatrix} R \end{bmatrix}$ and $\begin{bmatrix} R_2 \end{bmatrix}$ $\begin{bmatrix} \sigma_{\text{principal}} \end{bmatrix} = \begin{bmatrix} R_1 \end{bmatrix} \begin{bmatrix} \sigma \end{bmatrix} \begin{bmatrix} R_1 \end{bmatrix}^T = \begin{bmatrix} R_2 \end{bmatrix} \begin{bmatrix} \sigma_1 \end{bmatrix} \begin{bmatrix} R_2 \end{bmatrix}^T \begin{bmatrix} R_2 \end{bmatrix}^T \begin{bmatrix} R_1 \end{bmatrix} \begin{bmatrix} \sigma \end{bmatrix} \begin{bmatrix} R_1 \end{bmatrix}^T \begin{bmatrix} R_2 \end{bmatrix} = \begin{bmatrix} R_2 \end{bmatrix}^T \begin{bmatrix} R_2 \end{bmatrix} \begin{bmatrix} \sigma_1 \end{bmatrix} \begin{bmatrix} R_2 \end{bmatrix}^T \begin{bmatrix} R_2 \end{bmatrix} = \begin{bmatrix} \sigma_1 \end{bmatrix}$ Therefore: $\begin{bmatrix} R \end{bmatrix} = \begin{bmatrix} R_2 \end{bmatrix}^T \begin{bmatrix} R_1 \end{bmatrix}$, $\begin{bmatrix} R \end{bmatrix}^T = \begin{bmatrix} R_1 \end{bmatrix}^T \begin{bmatrix} R_2 \end{bmatrix}$

We find $[R_1]$ and $[R_2]$ as usual:

$$\begin{bmatrix} \sigma \end{bmatrix} = \begin{bmatrix} 1.0 & 1.73 & 1.0 \\ 1.73 & 0.75 & 0.433 \\ 1.0 & 0.433 & 0.25 \end{bmatrix} \Rightarrow \begin{array}{c} \sigma_1 = 3 \\ \sigma_2 = 0 \\ \sigma_3 = -1 \end{array} \Rightarrow \begin{array}{c} \mathbf{n} \leftrightarrow (0.707, 0.612, 0.36) \\ \mathbf{m} \leftrightarrow (0.0, 0.50, -0.866) \\ \mathbf{\hat{p}} \leftrightarrow (-0.707, 0.61, 0.36) \end{array}$$

$$\begin{bmatrix} \mathbf{R}_{1} \end{bmatrix} = \begin{bmatrix} 0.707 & 0.61 & 0.36 \\ 0.0 & 0.50 & -0.866 \\ -0.707 & 0.61 & 0.36 \end{bmatrix}$$
So,

$$\begin{bmatrix} \sigma' \end{bmatrix} = \begin{bmatrix} 0.50 & 1.414 & 0.50 \\ 1.414 & 1.0 & 1.414 \\ 0.50 & 1.414 & 0.50 \end{bmatrix} \implies \begin{array}{c} \sigma_1 = 3 \\ \sigma_2 = 0 \\ \sigma_1 = -1 \end{array} \implies \begin{array}{c} \widehat{\mathbf{n}} \leftrightarrow (0.50, 0.707, 0.50) \\ \widehat{\mathbf{m}} \leftrightarrow (0.707, 0, -0.707) \\ \widehat{\mathbf{p}} \leftrightarrow (-0.50, 0.707, -0.50) \end{array}$$

$$\begin{bmatrix} \mathbf{R}_{2} \end{bmatrix} = \begin{bmatrix} 0.50 & 0.707 & 0.50 \\ 0.707 & 0.0 & -0.707 \\ -0.50 & 0.707 & -0.50 \end{bmatrix} \begin{bmatrix} \mathbf{R} \end{bmatrix} \begin{bmatrix} \mathbf{R}_{2} \end{bmatrix}^{\mathrm{T}} \begin{bmatrix} \mathbf{R}_{1} \end{bmatrix} = \begin{bmatrix} 0.71 & 0.36 & -0.61 \\ 0 & 0.86 & 0.51 \\ 0.71 & -0.36 & 0.61 \end{bmatrix}$$

17. Imagine that we define a new measure of stress, [S], as a matrix of components relating force components to area components, but that the force components are defined in two ways: 1. in terms of a different coordinate system than the area components, or 2. the are transformed according to a fixed linear operation to represent a new vector in the same coordinate system. a) Is [S][symmetric? b) According to these two definitions, does [S] represent the components of tensor?

SOLUTION:

In either of cases 1 or 2, we note that the new force components (let us call these components g_i) may be obtained from the standard force components f_i as follows (note that by "standard" we mean the components of a force as normally defined in the same coordinate system used to

express vector area components):

$$[g] = [L][f]$$
, where $[L]$ is a linear operator (rotation or other transformation matrix)

a. The definition of [S] follows from the expression of [g]:

$$\begin{bmatrix} g \end{bmatrix} = \begin{bmatrix} s \end{bmatrix} \begin{bmatrix} a \end{bmatrix} \implies \begin{bmatrix} L \end{bmatrix} \begin{bmatrix} f \end{bmatrix} = \begin{bmatrix} s \end{bmatrix} \begin{bmatrix} a \end{bmatrix}, \implies \begin{bmatrix} f \end{bmatrix} = \underbrace{\begin{bmatrix} L \end{bmatrix}^{-i} \begin{bmatrix} s \end{bmatrix} \begin{bmatrix} a \end{bmatrix} \begin{bmatrix} \sigma \end{bmatrix}$$
$$\vdots \quad \begin{bmatrix} \sigma \end{bmatrix} = \begin{bmatrix} L \end{bmatrix}^{-i} \begin{bmatrix} s \end{bmatrix}, \text{ or } \quad \begin{bmatrix} s \end{bmatrix} = \begin{bmatrix} L \end{bmatrix} \begin{bmatrix} \sigma \end{bmatrix}$$

Since $[\mathcal{L}]$ is a general, non-symmetric matrix, [s] is in general not symmetric.

b. In order to examine how the new stress measure [S] transforms, let's imagine that we want to express [S] in a new coordinate system: $\begin{bmatrix} x' \end{bmatrix} = \begin{bmatrix} R \end{bmatrix} \begin{bmatrix} x \end{bmatrix}$. In the new coordinate system, our definitions will be expressed as follows:

$$[g'] = [S'][a']_{, where} [a'] = [R][a]$$

The central question, the one that differentiates Case I from Case II, is: What is the meaning of $\begin{bmatrix} g' \end{bmatrix}_2$

Case 1 - According to the definition of Case 1, $\begin{bmatrix} g' \end{bmatrix}$ is found in the new coordinate system by applying the fixed linear operator, $[\mathcal{L}]$, to the components of [f] in the new coordinate system:

 $\begin{bmatrix} g' \end{bmatrix} = \begin{bmatrix} L \end{bmatrix} \begin{bmatrix} f \end{bmatrix} = \begin{bmatrix} L \end{bmatrix} \begin{bmatrix} R \end{bmatrix} \begin{bmatrix} f \end{bmatrix},$ once this expression for $\begin{bmatrix} g' \end{bmatrix}$ is found, we can find how $\begin{bmatrix} S \end{bmatrix}$ and $\begin{bmatrix} S \end{bmatrix}$ are related: $\begin{bmatrix} g' \end{bmatrix} = \begin{bmatrix} L \end{bmatrix} \begin{bmatrix} R \end{bmatrix} \begin{bmatrix} f \end{bmatrix} = \begin{bmatrix} S \end{bmatrix} \begin{bmatrix} a' \end{bmatrix} = \begin{bmatrix} S' \end{bmatrix} \begin{bmatrix} R \end{bmatrix} \begin{bmatrix} a \end{bmatrix}$

$$[f] = [R]^{T}[L]^{-1}[S'][R][a], \text{ but note that } [f] = [L]^{-1}[g], \text{ so}$$
$$[g] = \underbrace{[L][R]^{T}[L]^{-1}[S'][R]}_{[S]}[a], \text{ and therefore}$$
$$[S] = [L][R]^{T}[L]^{-1}[S'][R], \text{ or } [S'] = [L]^{-1}[R][L][S][R]^{T}$$

Clearly this last expression is not the proper transformation for tensor components, so [S] defined as in Case 1 does not represent tensor components.

Case 2 - According to the definition of Case 2, $\begin{bmatrix} g' \end{bmatrix}$ is found in the new coordinate system by simply transforming the compents of g as any other vector in the original coordinate system, i.e. $\begin{bmatrix} g' \end{bmatrix} = \begin{bmatrix} R \end{bmatrix} \begin{bmatrix} g \end{bmatrix}$

once we have this expression for [g'], we proceed as before to find the relationship between [S] and [S]:

$$\begin{bmatrix} g' \end{bmatrix} = \begin{bmatrix} R \end{bmatrix} \begin{bmatrix} g \end{bmatrix} = \begin{bmatrix} S' \end{bmatrix} \begin{bmatrix} a \end{bmatrix} = \begin{bmatrix} S' \end{bmatrix} \begin{bmatrix} R \end{bmatrix} \begin{bmatrix} a \end{bmatrix}$$
$$\begin{bmatrix} g \end{bmatrix} = \underbrace{\begin{bmatrix} R \end{bmatrix}^{T} \begin{bmatrix} S' \end{bmatrix} \begin{bmatrix} R \end{bmatrix} \begin{bmatrix} a \end{bmatrix}$$
, and therefore

$$\begin{bmatrix} S \end{bmatrix} = \begin{bmatrix} R \end{bmatrix}^{T} \begin{bmatrix} S \end{bmatrix} \begin{bmatrix} R \end{bmatrix}, \text{ or } \begin{bmatrix} S' \end{bmatrix} = \begin{bmatrix} R \end{bmatrix} \begin{bmatrix} S \end{bmatrix} \begin{bmatrix} R \end{bmatrix}^{T}$$

This last expression is precisely the transformation for tensor components, so [S] defined in Case 2 does represent tensor components. Put more simply, S defined according to Case 2 is a proper tensor. In fact, representation of stress in this manner is convenient in some cases, where the force or area vectors are rotated to correspond to deformed or undeformed states in a material.

CHAPTER 4 - PROBLEM SOLUTIONS

A. PROFICIENCY PROBLEMS

 $\mathbf{F} \leftrightarrow \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix}$ Given:

1.

- a. Find E_{ij} , and ϵ_{ij} , the components of the large and small strain tensors, respectively.
- b. Using E and ε directly, find the new length of the vectors OA and AB shown below. Note that the original vectors are of unit length.



c. Why are the deformed lengths of O'A' and O'B' different when calculated using the two different measures of deformation?

SOLUTION:

We have from the definitions: $[C] = [F]^{T}[F] = \begin{bmatrix} 1 & 3 \\ 2 & 4 \end{bmatrix} \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix} = \begin{bmatrix} 10 & 14 \\ 14 & 20 \end{bmatrix}$ a.

so that the strain tensor E is: $[E] = \frac{1}{2} ([C] - [I]) = \begin{bmatrix} 4.5 & 7\\ 7 & 9.5 \end{bmatrix}$

while the small strain tensor $\mathbf{\varepsilon}$ is: $[\varepsilon] = \frac{1}{2} ([F]^{T} + [F]) - [I] = \begin{bmatrix} 0 & 2.5 \\ 2.5 & 3 \end{bmatrix}$

b. The change of length Δl of the OA vector is such that: $\Delta l_{A}^{2} = [OA]^{T} 2 [E] [OA] = \begin{bmatrix} 1 & 0 \end{bmatrix} 2 \begin{bmatrix} 4.5 & 7 \\ 7 & 9.5 \end{bmatrix} \begin{bmatrix} 1 \\ 0 \end{bmatrix} = 9$

The final length is then: $l_A = \sqrt{OA^2 + \Delta l_A^2} = \sqrt{10}$

For the OB vector the same approach gives:

$$\Delta l_{B}^{2} = [OB]^{T} 2 [E] [OB] = \begin{bmatrix} 0 & 1 \end{bmatrix} 2 \begin{bmatrix} 4.5 & 7 \\ 7 & 9.5 \end{bmatrix} \begin{bmatrix} 0 \\ 1 \end{bmatrix} = 19$$
with the final length: $l_{B} = \sqrt{OB^{2} + \Delta l_{B}^{2}} = \sqrt{20}$

Page 2

With the small strain tensor $[\varepsilon]$ the new lengths are:

$$l_{A}^{2} = 1 + \begin{bmatrix} OA \end{bmatrix}^{T} 2 \begin{bmatrix} \varepsilon \end{bmatrix} \begin{bmatrix} OA \end{bmatrix} = 1 + \begin{bmatrix} 1 & 0 \end{bmatrix} 2 \begin{bmatrix} 0 & 2.5 \\ 2.5 & 0 \end{bmatrix} \begin{bmatrix} 1 \\ 0 \end{bmatrix} = 1$$

and

$$l_{\rm B}^2 = 1 + \begin{bmatrix} OB \end{bmatrix}^{\rm T} 2 \begin{bmatrix} \varepsilon \end{bmatrix} \begin{bmatrix} OB \end{bmatrix} = 1 + \begin{bmatrix} 0 & 1 \end{bmatrix} 2 \begin{bmatrix} 0 & 2.5 \\ 2.5 & 3 \end{bmatrix} \begin{bmatrix} 0 \\ 1 \end{bmatrix} = 7$$

- c. The results are quite different when we use the two strain measures: in fact the use of the $[\epsilon]$ tensor is not valid here as the strain components are not much less than 1, as required for accuracy.
- 2. Given the figure below for an assumed homogeneous deformation, write down the deformation gradient, F:



SOLUTION:

The unknown deformation gradient is denoted by: $\begin{bmatrix} a & b \\ c & d \end{bmatrix} \begin{bmatrix} 2 \\ 0 \end{bmatrix} = \begin{bmatrix} 8.1 & -6 \\ 3 & -3 \end{bmatrix} \Rightarrow \begin{bmatrix} 2 & a \\ 2 & c \end{bmatrix} = \begin{bmatrix} 2.1 \\ 0 \end{bmatrix}$ and $\begin{bmatrix} a & b \\ c & d \end{bmatrix} \begin{bmatrix} 0 \\ 2 \end{bmatrix} = \begin{bmatrix} 6.6 & -6 \\ 4.6 & -3 \end{bmatrix} \Rightarrow \begin{bmatrix} 2 & b \\ 2 & d \end{bmatrix} = \begin{bmatrix} 0.6 \\ 1.6 \end{bmatrix}$ we conclude that: $\begin{bmatrix} F \end{bmatrix} = \begin{bmatrix} 1.05 & 0.3 \\ 0 & 0.8 \end{bmatrix}$ we conclude that: $\begin{bmatrix} F \end{bmatrix} = \begin{bmatrix} 0.1 & 0.2 & 0.5 \\ 0.3 & 0.4 & 0.6 \\ 0.7 & 0.8 & 0.9 \end{bmatrix}$

Find: J, E, and ε .

SOLUTION:

3.

From the equality: [J] = [F] - [I] we get: $[J] = \begin{bmatrix} -0.9 & 0.2 & 0.5 \\ 0.3 & -0.6 & 0.6 \\ 0.7 & 0.8 & -0.1 \end{bmatrix}$ The strain tensor [E] is writen:

$$\begin{split} [\mathbf{E}] &= \frac{1}{2} \left([\mathbf{F}]^{\mathrm{T}} [\mathbf{F}] - [\mathbf{I}] \right) = \frac{1}{2} \left(\begin{bmatrix} 0.1 & 0.3 & 0.7 \\ 0.2 & 0.4 & 0.8 \\ 0.5 & 0.6 & 0.9 \end{bmatrix} \begin{bmatrix} 0.1 & 0.2 & 0.5 \\ 0.3 & 0.4 & 0.6 \\ 0.7 & 0.8 & 0.9 \end{bmatrix} - \begin{bmatrix} 1.0 & 0 & 0 \\ 0 & 1.0 & 0 \\ 0 & 0 & 1.0 \end{bmatrix} \right) \\ &= \frac{1}{200} \begin{bmatrix} 1+3*3+7*7-100 & 2+3*4+7*8 & 5+3*6+7*9 \\ \text{sym} & 2*2+4*4+8*8-100 & 2*5+4*6+8*9 \\ \text{sym} & \text{sym} & 5*5+6*6+9*9-100 \end{bmatrix} \\ &= \begin{bmatrix} -0.205 & 0.35 & 0.43 \\ 0.35 & -0.08 & 0.53 \\ 0.43 & 0.53 & 0.21 \end{bmatrix} \end{split}$$

The small strain tensor is:

$$[\varepsilon] = \frac{1}{2} \left(\begin{bmatrix} -0.9 & 0.2 & 0.5 \\ 0.3 & -0.6 & 0.6 \\ 0.7 & 0.8 & -0.1 \end{bmatrix} + \left[\begin{array}{ccc} -0.9 & 0.3 & 0.7 \\ 0.2 & -0.6 & 0.8 \\ 0.5 & 0.6 & -0.1 \end{bmatrix} \right) = \left[\begin{array}{ccc} -0.9 & 0.25 & 0.6 \\ 0.25 & -0.6 & 0.7 \\ 0.6 & 0.7 & -0.1 \end{bmatrix} \right)$$

4. As shown below, a point in a continuum (O) moves to a new point (O') as shown.



a. Find the new points A' and B', assuming homogeneous deformation for the following two cases:



b. For each deformation, find **E** the large strain tensor.

SOLUTION:

By definition of the deformation gradient (when there is an homogeneous deformation), we obtain for the first case:

$$OA \rightarrow O'A' \text{ with: } \left[O'A'\right] = [F][OA] = \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix} \begin{bmatrix} 0 \\ 1.5 \end{bmatrix} = \begin{bmatrix} 3 \\ 6 \end{bmatrix}$$

and
$$OB \rightarrow O'B' \text{ with: } \left[O'B'\right] = [F][OB] = \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix} \begin{bmatrix} 1.5 \\ 0 \end{bmatrix} = \begin{bmatrix} 1.5 \\ 4.5 \end{bmatrix}$$

$$\begin{bmatrix} OA' \end{bmatrix} = \begin{bmatrix} OO' \end{bmatrix} + \begin{bmatrix} O'A' \end{bmatrix} = \begin{bmatrix} 4 \\ 2 \end{bmatrix} + \begin{bmatrix} 3 \\ 6 \end{bmatrix} = \begin{bmatrix} 7 \\ 8 \end{bmatrix}$$
$$\begin{bmatrix} OB' \end{bmatrix} = \begin{bmatrix} OO' \end{bmatrix} + \begin{bmatrix} O'B' \end{bmatrix} = \begin{bmatrix} 4 \\ 2 \end{bmatrix} + \begin{bmatrix} 1.5 \\ 4.5 \end{bmatrix} = \begin{bmatrix} 5.5 \\ 6.5 \end{bmatrix}$$

so that:

$$[\mathbf{F}] = [\mathbf{J}] + [\mathbf{I}] = \begin{bmatrix} 3 & 1 \\ 4 & 4 \end{bmatrix}$$

For the second case we have first:

The transformation verifies:

$$OA \rightarrow O'A' \text{ with: } \left[O'A'\right] = [F][OA] = \begin{bmatrix} 3 & 1 \\ 4 & 4 \end{bmatrix} \begin{bmatrix} 0 \\ 1.5 \end{bmatrix} = \begin{bmatrix} 1.5 \\ 6 \end{bmatrix}$$

$$OB \rightarrow O'B' \text{ with: } \left[O'B'\right] = [F][OB] = \begin{bmatrix} 3 & 1 \\ 4 & 4 \end{bmatrix} \begin{bmatrix} 1.5 \\ 0 \end{bmatrix} = \begin{bmatrix} 4.5 \\ 6 \end{bmatrix}$$

$$\left[OA'\right] = \left[OO'\right] + \left[O'A'\right] = \begin{bmatrix} 4 \\ 2 \end{bmatrix} + \begin{bmatrix} 1.5 \\ 6 \end{bmatrix} = \begin{bmatrix} 5.5 \\ 8 \end{bmatrix}$$

$$\left[OB'\right] = \left[OO'\right] + \left[O'B'\right] = \begin{bmatrix} 4 \\ 2 \end{bmatrix} + \begin{bmatrix} 4.5 \\ 6 \end{bmatrix} = \begin{bmatrix} 5.5 \\ 8 \end{bmatrix}$$
so that

b. For the first case we have

$$[\mathbf{E}] = \frac{1}{2} \left([\mathbf{F}]^{\mathrm{T}} [\mathbf{F}] - [\mathbf{I}] \right) = \frac{1}{2} \left(\begin{bmatrix} 1 & 3 \\ 2 & 4 \end{bmatrix} \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix} - \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \right) = \begin{bmatrix} 4.5 & 7 \\ 7 & 9.5 \end{bmatrix}$$

and for the second case

$$\begin{bmatrix} \mathbf{E} \end{bmatrix} = \frac{1}{2} \left(\begin{bmatrix} \mathbf{J} \end{bmatrix} + \begin{bmatrix} \mathbf{J} \end{bmatrix}^{\mathrm{T}} + \begin{bmatrix} \mathbf{J} \end{bmatrix}^{\mathrm{T}} \begin{bmatrix} \mathbf{J} \end{bmatrix} \right) = \frac{1}{2} \left(\begin{bmatrix} 2 & 1 \\ 4 & 3 \end{bmatrix} + \begin{bmatrix} 2 & 4 \\ 1 & 3 \end{bmatrix} + \begin{bmatrix} 2 & 4 \\ 1 & 3 \end{bmatrix} \begin{bmatrix} 2 & 1 \\ 4 & 3 \end{bmatrix} \right) = \begin{bmatrix} 12 & 9.5 \\ 9.5 & 8.0 \end{bmatrix}$$

5. Imagine that a line segment OP is embedded in a material which is deformed to a new state. The line segment becomes O'P' after deformation, as shown below.



1

- a. Find the vector components of $\overline{O'P'}$:
- b. Find the length of $\overline{O'P'}$ if:

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c. Find the components of \overline{OP} if:



SOLUTION:

- a. If the deformation is homogeneous we can write: $OP \rightarrow O'P'$ such that: $[O'P'] = [F][OP] = \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix} \begin{bmatrix} 2 \\ 1 \end{bmatrix} = \begin{bmatrix} 4 \\ 10 \end{bmatrix}$
- b. According to Eq. 4.19: $O'P'^{2} = [OP]^{T}[C][OP] = \begin{bmatrix} 2 & 1 \end{bmatrix} \begin{bmatrix} 1 & 3 \\ 3 & 2 \end{bmatrix} \begin{bmatrix} 2 \\ 1 \end{bmatrix} = \begin{bmatrix} 2 & 1 \end{bmatrix} \begin{bmatrix} 5 \\ 8 \end{bmatrix} = 18$

the final length is then: $O'P' = \sqrt{18} \approx 4.24$

c. From [J] we deduce:

 $OP \rightarrow O'P'$ such that: $[O'P'] = ([J] + [I])[OP] = \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix} \begin{bmatrix} 2 \\ 1 \end{bmatrix} = \begin{bmatrix} 4 \\ 10 \end{bmatrix}$

6. A homogeneous deformation is imposed in the plane of the sheet. Two lines painted on the surface move as shown below, with coordinates measured as shown:



- a. Find F, the deformation gradient.
- b. Starting with F, find C, E, and ε .
- c. Find the principal strains and axes of **E**.

SOLUTION:

- a. The method is similar to that used for Problem 2. We express that the deformation gradient applied to the two vectors (as rows) gives their transformed coordinates:
 - $\begin{bmatrix} a & b \\ c & d \end{bmatrix} \begin{bmatrix} 1 & 0.8 \\ 0.5 & 1.1 \end{bmatrix} = \begin{bmatrix} 1.1 & 0.1 \\ -0.1 & 1.3 \end{bmatrix}$

The best method is to invert the matrix on the left hand side: $\begin{bmatrix} 1 & 0.8\\ 0.5 & 1.1 \end{bmatrix}^{-1} = \frac{1}{0.7} \begin{bmatrix} 1.1 & -0.8\\ -0.5 & 1 \end{bmatrix}$

so that the unknown deformation gradient is:

 $\begin{bmatrix} a & b \\ c & d \end{bmatrix} = \begin{bmatrix} 1.1 & 0.1 \\ -0.1 & 1.3 \end{bmatrix} \frac{1}{0.7} \begin{bmatrix} 1.1 & -0.8 \\ -0.5 & 1 \end{bmatrix} = \frac{1}{0.7} \begin{bmatrix} 1.16 & -0.78 \\ -0.76 & 1.38 \end{bmatrix} \cong \begin{bmatrix} 1.66 & -1.11 \\ -1.09 & 1.97 \end{bmatrix}$

b. The usual tensors are computed:

$$\begin{bmatrix} C \end{bmatrix} = \frac{1}{0.7} \begin{bmatrix} 1.16 & -0.76 \\ -0.78 & 1.38 \end{bmatrix} \frac{1}{0.7} \begin{bmatrix} 1.16 & -0.78 \\ -0.76 & 1.38 \end{bmatrix} = \frac{1}{0.49} \begin{bmatrix} 1.9232 & -1.9536 \\ -1.9536 & 2.5128 \end{bmatrix} \cong \begin{bmatrix} 3.92 & -3.99 \\ -3.99 & 5.13 \end{bmatrix}$$
$$\begin{bmatrix} E \end{bmatrix} = \frac{1}{2} \begin{bmatrix} 3.92 & -1 & -3.99 \\ -3.99 & 5.13 & -1 \end{bmatrix} = \begin{bmatrix} 1.46 & -2 \\ -2 & 2.07 \end{bmatrix}$$
$$\begin{bmatrix} E \end{bmatrix} = \frac{1}{2} \begin{bmatrix} 1.66 & -1 & -1.11 \\ -1.09 & 1.97 & -1 \end{bmatrix} + \begin{bmatrix} 1.66 & -1 & -1.09 \\ -1.11 & 1.97 & -1 \end{bmatrix} = \begin{bmatrix} 0.66 & -1.10 \\ -1.10 & 0.97 \end{bmatrix}$$

(But again in this case the small strain tensor has no precise meaning, as its components are not small with respect to 1).

c. The principal strains are solutions of the eigenvalue problem:

$$\begin{array}{ccc} 1.46 - \lambda & -2 \\ -2 & 2.07 - \lambda \end{array} = 0, \text{ or } (1.46 - \lambda) (2.07 - \lambda) - 4 = 0 \end{array}$$

which is also:

$$\lambda^2 - 3.53 \ \lambda - 0.98 = 0$$
 with solutions $E_1 = -0.26$ and $E_2 = 3.79$

The principal axes \mathbf{v}_1 and \mathbf{v}_2 are such that their components verify:

$$(1.46 + 0.26) v_{11} - 2 v_{21} = 0 \implies [v_1] = \begin{bmatrix} 0.76 \\ 0.65 \end{bmatrix}$$
$$(1.46 - 3.79) v_{12} - 2 v_{22} = 0 \implies [v_2] = \begin{bmatrix} -0.65 \\ 0.76 \end{bmatrix}$$

7. At time t, the position of a material particle initially at (X_1, X_2, X_3) is

$$x_1 = X_1 + a X_2$$
$$x_2 = X_2 + a X_1$$
$$x_3 = X_3$$

Obtain the **unit elongation** (*i. e. change in length per unit initial length*) *of an element initially in the direction of* $\hat{\mathbf{x}}_1 + \hat{\mathbf{x}}_2$.

SOLUTION:

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$$\mathbf{F} = \frac{\check{\mathbf{Z}}\mathbf{x}}{\check{\mathbf{Z}}\mathbf{X}}, \ [\mathbf{F}] = \begin{bmatrix} 1 & a & 0 \\ a & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

The deformation gradient is here:

$$[C] = \begin{bmatrix} 1+a^2 & 2a & 0\\ 2a & 1+a^2 & 0\\ 0 & 0 & 1 \end{bmatrix}$$

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and the Cauchy strain tensor is immediately deduced: $\begin{bmatrix} 0 & 0 \end{bmatrix}$

$$\mathrm{d}\mathbf{S} \leftrightarrow \begin{bmatrix} \frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} \\ 0 \end{bmatrix}$$

The initial unit vector is:

The length of the final corresponding vector verifies:

$$ds^{2} = [dS]^{T}[C][dS] = \begin{bmatrix} \frac{1}{\sqrt{2}} & 0 \end{bmatrix} dS \begin{bmatrix} 1+a^{2} & 2a & 0 \\ 2a & 1+a^{2} & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} \frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} \\ 0 \end{bmatrix} dS = (1+a)^{2} dS^{2}$$

so that ds = (1 + a) dS and the unit elongation is: E = a.

- 8. Take fixed right handed axes \mathbf{x}_1 , \mathbf{x}_2 , \mathbf{x}_3 . Write down the deformation gradient matrix, $\frac{\partial X_i}{\partial x_j}$, for the deformation of a body from \mathbf{x} to \mathbf{X} for
 - a. right handed rotation of 45° about $\mathbf{\hat{x}}_{1}$.
 - b. Left handed rotation of 45° about $\widehat{\mathbf{x}}_{2}$.
 - c. Stretch by a stretch ratio of 2 in the $\hat{\mathbf{x}}_3$ direction.
 - *d.* Stretch by a stretch ratio of $\frac{1}{2}$ in the $\hat{\mathbf{x}}_2$ direction.
 - e. Right handed rotation of 90° about $\hat{\mathbf{x}}_{3}$.

Find the total deformation matrix for these motions carried out sequentially. Using this result, check the final volume ratio.

SOLUTION:

The deformation gradients are computed at each step:

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$$\begin{bmatrix} F_{a} \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & \frac{1}{1/2} & -\frac{1}{1/2} \\ 0 & \frac{1}{1/2} & \frac{1}{1/2} \end{bmatrix}, \quad \begin{bmatrix} F_{b} \end{bmatrix} = \begin{bmatrix} \frac{1}{1/2} & 0 & -\frac{1}{1/2} \\ 0 & 1 & 0 \\ \frac{1}{1/2} & 0 & \frac{1}{1/2} \end{bmatrix}$$
$$\begin{bmatrix} F_{c} \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 2 \end{bmatrix}, \quad \begin{bmatrix} F_{d} \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1/2 & 0 \\ 0 & 0 & 1 \end{bmatrix}, \quad \begin{bmatrix} F_{e} \end{bmatrix} = \begin{bmatrix} 0 & -1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

The combined transformation has a deformation gradient: $\begin{bmatrix} 0 & \sqrt{2} & \sqrt{2} \end{bmatrix}$

$$[F] = [F_e][F_d][F_c][F_b][F_a] = \begin{bmatrix} 0 & -\frac{12}{4} & \frac{12}{4} \\ \frac{\sqrt{2}}{2} & -\frac{1}{2} & -\frac{1}{2} \\ \frac{\sqrt{2}}{2} & 1 & 1 \end{bmatrix}$$

$$\begin{bmatrix} C \end{bmatrix} = \begin{bmatrix} 2.5 & 1.06 & 1.06 \\ 1.06 & 1.38 & 1.12 \\ 1.06 & 1.12 & 1.38 \end{bmatrix}$$

The Cauchy strain tensor is then evaluated:

We see immediately that: $|\mathbf{Q} = 1$, i. e. the deformation takes place with no volume change.

9. From the following mapping, find C, U, and R: $\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 2 & 0 & 0 \\ 0 & 2 & 4 \end{bmatrix} \begin{bmatrix} X_1 \\ X_2 \end{bmatrix}$

x ₁		2	0	0	X_1
x ₂	=	0	3	4	X_2
x3_			4	-3	X_3

Check whether this is a permissible deformation in a continuous body.

SOLUTION:

The Cauchy strain tensor is first calculated by:

$$\begin{bmatrix} \mathbf{C} \end{bmatrix} = \begin{bmatrix} 2 & 0 & 0 \\ 0 & 3 & 4 \\ 0 & 4 & -3 \end{bmatrix} \begin{bmatrix} 2 & 0 & 0 \\ 0 & 3 & 4 \\ 0 & 4 & -3 \end{bmatrix} = \begin{bmatrix} 4 & 0 & 0 \\ 0 & 25 & 0 \\ 0 & 0 & 25 \end{bmatrix}$$

The polar decomposition is written: [F] = [R] [U]; with $[U]^2 = [C]$ we deduce:

$$\begin{bmatrix} U \end{bmatrix} = \begin{bmatrix} 2 & 0 & 0 \\ 0 & 5 & 0 \\ 0 & 0 & 5 \end{bmatrix} \text{ and } \begin{bmatrix} R \end{bmatrix} = \begin{bmatrix} F \end{bmatrix} \begin{bmatrix} U \end{bmatrix}^{-1} = \begin{bmatrix} 2 & 0 & 0 \\ 0 & 3 & 4 \\ 0 & 4 & -3 \end{bmatrix} \begin{bmatrix} \frac{1}{2} & 0 & 0 \\ 0 & \frac{1}{5} & 0 \\ 0 & 0 & \frac{1}{5} \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & \frac{3}{5} & \frac{4}{5} \\ 0 & \frac{4}{5} & -\frac{3}{5} \end{bmatrix}$$

But we see that $|\mathbf{R}| = -1$ so that $[\mathbf{R}]$ is not a rotation, but rather an inversion.

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0

0

10. Check the compatibility of the following strain components:

SOLUTION:

$$\begin{bmatrix} \varepsilon \end{bmatrix} = \begin{bmatrix} x_1 + x_2 & x_1 + 2x_2 & 0 \\ x_1 + 2x_2 & x_1 + x_2 & 0 \\ 0 & 0 & x_3 \end{bmatrix}$$

From the strain tensor:

the components of which are first order polynomials, we see that: (see also Problem 15)

$$\frac{\check{Z}^2 \varepsilon_{ii}}{\check{Z} x_j^2} + \frac{\check{Z}^2 \varepsilon_{jj}}{\check{Z} x_i^2} = 2 \frac{\check{Z}^2 \varepsilon_{ij}}{\check{Z} x_i \partial x_j} = 0$$

so that the field is compatible.

$$\begin{bmatrix} \varepsilon \end{bmatrix} = \begin{bmatrix} x_1^2 & x_1^2 + x_2^2 & x_2 x_3 \\ x_1^2 + x_2^2 & x_2^2 & 0 \\ x_2 x_3 & 0 & 0 \end{bmatrix}$$

The second strain tensor is:

we observe that:
$$\frac{\check{Z}^2 \varepsilon_2}{\check{Z} x_j^2} + \frac{\check{Z}^2 \varepsilon_{jj}}{\check{Z} x_i^2} = 2 \frac{\check{Z}^2 \varepsilon_{ij}}{\check{Z} x_1 \partial x_2} = 0$$

 $\tilde{Z}x_3^2$

$$\frac{\check{Z}^2 \varepsilon_{22}}{\check{Z} x_3^2} + \frac{\check{Z}^2 \varepsilon_{33}}{\check{Z} x_2^2} = 0 = 2 \frac{\check{Z}^2 \varepsilon_{23}}{\check{Z} x_2 \partial x_3}$$
$$\frac{\check{Z}^2 \varepsilon_{33}}{\check{Z} x_2 \partial x_3} + \frac{\check{Z}^2 \varepsilon_{11}}{\check{Z} x_2 \partial x_3} = 0 = 2 \frac{\check{Z}^2 \varepsilon_{13}}{\check{Z} x_2 \partial x_3}$$

B. DEPTH PROBLEMS

11. Consider the extension of an arbitrary small line element AB. Start by examining how $(A'B')^2$ is related to $(AB)^2$ using the small extensional strain along that direction, e_n . Show that for small strains and displacements, rotations do not cause extension, i.e. if extensions are zero, strains are zero.

 $\check{Z}x_1 \partial x_3$

SOLUTION:

We put: AB = a dl, where a is a unit vector, and write Eq. 4.18 in the form:

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$$A'B'^{2} = (\mathbf{a} dl)^{T} \mathbf{C} \mathbf{a} dl = dl^{2} \mathbf{a}^{T} (\mathbf{I} + \mathbf{J})^{T} (\mathbf{I} + \mathbf{J}) \mathbf{a} = AB^{2} \mathbf{a}^{T} (\mathbf{I} + \mathbf{J}^{T} + \mathbf{J} + \mathbf{J}^{T} \mathbf{J}) \mathbf{a}$$

Using the small strain approximation we obtain:

 $A'B'^2 \cong AB^2 \mathbf{a}^T (\mathbf{I} + \mathbf{J}^T + \mathbf{J}) \mathbf{a} = AB^2 \mathbf{a}^T (\mathbf{I} + 2 \boldsymbol{\epsilon}) \mathbf{a}$

As the components of the strain tensor are small with respect to unity we can also write: $A'B' \cong AB (1 + 2 \mathbf{a}^T \mathbf{\epsilon} \mathbf{a})^{1/2} \cong AB (1 + \mathbf{a}^T \mathbf{\epsilon} \mathbf{a})$

If we have only an extension e_n in the x_1 direction, the previous equation becomes:

 $A'B' \cong AB(1 + e_n)$ (where we used $\begin{bmatrix} a \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$)

If there is zero extension in any direction we have:

$$\mathbf{A'B'} = \mathbf{AB} = \mathbf{AB} (1 + \mathbf{a}^{\mathrm{T}} \mathbf{\epsilon} \mathbf{a})$$

for any a, that is also:

 $\mathbf{a}^{\mathrm{T}} \mathbf{\epsilon} \mathbf{a} = 0$

If **a** is a principal direction, the above equality shows that the corresponding principal strain must be zero, so that we come to the conclusion that the strain tensor must be null.

12. In sheet forming, one often measures strains from a grid on the sheet surface. Then, one can plot these strains as a function of the original length along an originally straight line:



As shown in the second figure (for a simple forming operation), this originally straight line is curved and stretched.

If the edges of the sheet do not move (stretch boundary conditions), develop a rule that the measured strain distribution must follow. Consider that the original sheet length l_o becomes l at some later time.

SOLUTION:

We consider the strain ε_1 or e_1 (true or engineering definition) corresponding to the elongation in the initial direction of the line. A small vector dl_0 on the initial line will be transformed into a

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small vector dl so that its new length verifies: $dl = exp(\varepsilon_1) dl_0 = (1 + e_1) dl_0$. Therefore the final length of the line will be:

$$l = \int_0^{l_0} \exp(\varepsilon_l) \, dl_0 = \int_0^{l_0} (1 + e_1) \, dl_0$$

13. Consider a 1" square of material deformed in the following ways:



Find F for each case.
Find C and E.
Find the principal values and directions of E.
Find the material principal directions after deformation.
Which of these cases (a-d) are mechanically the same under isotropic conditions? Under general anisotropy?

SOLUTION:

a. We have: $[F] = \begin{bmatrix} 3 & 0 \\ 0 & 2 \end{bmatrix}$, $[C] = \begin{bmatrix} 9 & 0 \\ 0 & 4 \end{bmatrix}$, $[E] = \begin{bmatrix} 4 & 0 \\ 0 & 3/2 \end{bmatrix}$

The principal strains are obviously: $E_1 = 3/2$, $E_2 = 4$, the principal axes and the material axes after deformation are the ox₁ and ox₂ axes.

b. The new vectors are projected on the ox_1 and ox_2 axes so that we have:

$$[F] = \begin{bmatrix} 3\sqrt{3}/2 & -1 \\ 3/2 & \sqrt{3} \end{bmatrix} , \quad [C] = \begin{bmatrix} 3\sqrt{3}/2 & 3/2 \\ -1 & \sqrt{3} \end{bmatrix} \begin{bmatrix} 3\sqrt{3}/2 & -1 \\ 3/2 & \sqrt{3} \end{bmatrix} = \begin{bmatrix} 9 & 0 \\ 0 & 4 \end{bmatrix} , \quad [E] = \begin{bmatrix} 4 & 0 \\ 0 & 3/2 \end{bmatrix}$$

The remaining is the same as case a.

c.
$$[F] = \begin{bmatrix} -3 & 0 \\ 0 & -2 \end{bmatrix}$$
, $[C] = \begin{bmatrix} 9 & 0 \\ 0 & 4 \end{bmatrix}$, $[E] = \begin{bmatrix} 4 & 0 \\ 0 & 3/2 \end{bmatrix}$

The remaining is the same as case a.

$$[F] = \begin{bmatrix} 3\sqrt{2}/2 & -\sqrt{2} \\ 3\sqrt{2}/2 & \sqrt{2} \end{bmatrix}, \quad [C] = \begin{bmatrix} 3\sqrt{2}/2 & 3\sqrt{2}/2 \\ -\sqrt{2} & \sqrt{2} \end{bmatrix} \begin{bmatrix} 3\sqrt{2}/2 & -\sqrt{2} \\ 3\sqrt{2}/2 & \sqrt{2} \end{bmatrix} = \begin{bmatrix} 9 & 0 \\ 0 & 4 \end{bmatrix}, \quad [E] = \begin{bmatrix} 4 & 0 \\ 0 & 3/2 \end{bmatrix}$$

The remaining is the same as case a.

For isotropic materials the four cases a-d are equivalent by definition, as a rotation has no effect on the material properties. Under general anisotropy case a is equivalent to the undeformed configuration (if the deformation does not introduce any change in the anisotropic coefficients), while cases b, c, d correspond to rotations of 30° ; 180° and 45° respectively, and the material properties will be rotated by the same rotation.

14. Given that
$$\begin{bmatrix} \mathbf{C} \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 2 \end{bmatrix}$$
, find \mathbf{F} when

- a. the principal material axes do not rotate, and
- b. the principal material axes rotate by 30° counterclockwise. Express your answers in the original coordinate system.

SOLUTION:

a. We use the polar decomposition: $\mathbf{F} = \mathbf{R}\mathbf{U}$, if there is no rotation, then $\mathbf{R} = \mathbf{I}$, $\mathbf{F} = \mathbf{U}$ is symmetric and $\mathbf{C} = \mathbf{F}\mathbf{F}$ so that here \mathbf{F} is easily computed:

$$[\mathbf{F}] = [\mathbf{U}] = \begin{bmatrix} 1 & 0 \\ 0 & \sqrt{2} \end{bmatrix}$$

$$[\mathbf{R}] = \begin{bmatrix} \frac{\sqrt{3}}{2} & -\frac{1}{2} \\ \frac{1}{2} & \frac{\sqrt{3}}{2} \end{bmatrix}$$

b. The rotation matrix is:

The [U] stretch tensor is the same and the deformation gradient is:

$$[F] = [R][U] = \begin{bmatrix} \frac{\sqrt{3}}{2} & -\frac{1}{2} \\ \frac{1}{2} & \frac{\sqrt{3}}{2} \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 0 & \sqrt{2} \end{bmatrix} = \begin{bmatrix} \frac{\sqrt{3}}{2} & -\frac{\sqrt{2}}{2} \\ \frac{1}{2} & \frac{\sqrt{6}}{2} \end{bmatrix}$$

15. Derive a set of compatibility equations corresponding to Eq. 4.52 for the three-dimensional case.

SOLUTION:

A non diagonal (small) strain component is calculated according to:

$$\epsilon_{ij} = \frac{1}{2} \left(\frac{ZU_i}{\check{Z}x_j} + \frac{ZU_j}{\check{Z}x_i} \right) \text{ for } i \circ j$$

so that we obtain:

Chapter 4

$$2 \frac{\check{Z}^2 \varepsilon_{ij}}{\check{Z} x_i \check{Z} x_j} = \frac{\check{Z}^3 U_i}{\check{Z} x_i \check{Z} x_j^2} + \frac{\check{Z}^3 U_j}{\check{Z} x_i^2 \check{Z} x_j} \quad \text{for} \quad i \, {}^\circ j$$

which is also, with the definition of the diagonal terms (without summation on i and j):

$$2\frac{\check{Z}^{2}\varepsilon_{ij}}{\check{Z}x_{i}\check{Z}x_{j}} = \frac{\check{Z}^{2}\varepsilon_{ii}}{\check{Z}x_{j}^{2}} + \frac{\check{Z}^{2}\varepsilon_{jj}}{\check{Z}x_{i}^{2}}$$

CHAPTER 5 - PROBLEM SOLUTIONS

A. PROFICIENCY PROBLEMS

1. Adapt the demonstration of the Green theorem of Eq. 5.1 to:

- a curve C, and show that: $\int_{C} f \frac{dg}{dl} dl = f(B) g(B) - f(A) g(A) - \int_{C} \frac{df}{dl} g dl$

where A and B are the extremities of the curve. Examine the case when the curve is closed (A=B).

- a surface S defined in the plane (with coordinates x_1 and x_2), so that:

$$\int_{S} f \frac{\check{Z}g}{\check{Z}x_{i}} dS = \int_{\check{Z}S} f g n_{i} dI - \int_{S} \frac{\check{Z}f}{\check{Z}x_{i}} g dS$$

i = 1 or 2, ∂S is the curve limiting the surface S (with no hole), and **n** is the unit normal vector to ∂S .

SOLUTION

We suppose that the curve is represented by parametric equations of the form: $x_i = x_i(1)$ for $0 \le 1 \le L$, where 1 is the arc lenght measured from the origine A of the curve. f and g can be considered as functions of the arc lenght 1 on the curve and will be denoted as: $F(1) = f(x_i(1), x_i(1)), and F(1) = g(x_i(1), x_i(1)), and F(1) = f(x_i(1), x_i(1))$

 $F(l) = f(x_1(l), x_2(l), x_3(l))$ and $G(l) = g(x_1(l), x_2(l), x_3(l))$

so that the left hand side of the 1-D equivalent of the Green equation becomes:

$$\int_C f \frac{dg}{dl} dl = \int_0^L F(l) \frac{dG(l)}{dl} dl$$

Using a general property of integrals of functions of one variable, we can write:

$$\int_{0}^{L} \frac{d}{dl} \left[G(l) F(l) \right] dl = F(L) G(L) - F(0) G(0) = \int_{0}^{L} \frac{dF(l)}{dl} G(l) dl + \int_{0}^{L} F(l) \frac{dG(l)}{dl} dl$$

If this equation is combined with the previous one, we get:

$$\int_0^L F(l) \frac{dG(l)}{dl} dl = F(L) G(L) - F(0) G(0) - \int_0^L \frac{dF(l)}{dl} G(l) dl$$

which is the desired equality when we come back to the initial notations.

The proof for the 2-D Green formula is very similar to the proof for the 3-D case. The surface S is represented in the figure below.



The boundary of S is divided into an upper part $\partial \Omega_U$, and a lower part $\partial \Omega_L$ the equations of which are respectively:

 $x_2 = h_U(x_1)$; $x_2 = h_L(x_1)$

$$\int_{\Omega} f \frac{\check{Z}g}{\check{Z}x_2} dS = \int_{x_A}^{x_B} \left(\int_{h_L(x_1)}^{h_U(x_1)} f \frac{\check{Z}g}{\check{Z}x_2} dx_2 \right) dx_1$$

Using these notations, we can write:

The above equation is transformed in a similar way as Eq. 5.3 so that:

$$\int_{h_L}^{h_U} f \frac{\check{Z}g}{\check{Z}x_2} dx_2 = f_U g_U - f_L g_L - \int_{h_L}^{h_U} \frac{\check{Z}f}{\check{Z}x_2} g dx_2$$

This equation is now integrated with respect to x₁:

$$\int_{S} f \frac{\check{Z}g}{\check{Z}x_{2}} dS = \int_{x_{A}}^{x_{B}} (f_{U} g_{U} - f_{L} g_{L}) dx_{1} - \int_{x_{A}}^{x_{B}} \left(\int_{h_{L}}^{h_{U}} \frac{\check{Z}f}{\check{Z}x_{2}} g dx_{2} \right) dx_{1}$$

In a way similar to Eqs. 5.3a and 5.3b, the component n_{U2} of the normal to $\partial \Omega_U$ is such that $dx_1 = n_{U2} dl$ and an analogous equation for the lower part of the boundary. Then we get:

$$\int_{x_A}^{x_B} (f_U g_U - f_L g_L) dx_1 = \int_{x_A}^{x_B} f_U g_U n_{U2} dl - \int_{x_A}^{x_B} f_L g_L n_{L2} dl = \int_{\partial S} fg dl$$

When this last result is put into the previous equation we obtain the 2-D form of the Green theorem:

$$\int_{S} f \frac{Zg}{Zx_2} dS = \int_{\partial S} fg n_2 dI - \int_{S} \frac{Zf}{Zx_2} g dS$$

2. Apply the 2-D form of the Green theorem (see problem 1) to the functions $f = x_1 + x_2$ and $g = x_1 - x_2$ in the square domain $[0,1]^2$, for i = 1 and for i = 2. Compute directly the integrals and verify the Green theorem on this specific example.

We must show that the following equality holds:

$$\int_0^1 \left(\int_0^1 f \frac{\check{Z}g}{\check{Z}x_2} \, dx_2 \right) dx_1 = \int_{\partial S} fg \, n_2 \, dl - \int_0^1 \left(\int_0^1 \frac{\check{Z}f}{\check{Z}x_2} \, g \, dx_2 \right) dx_1$$

The first integral in the above equation is computed according to:

$$\int_{0}^{1} \left(\int_{0}^{1} (x_{1} + x_{2}) (-1) dx_{2} \right) dx_{1} = \oint_{0}^{1} \left[x_{1}x_{2} + \frac{x_{1}^{2}}{2} \right]_{x_{2}=0}^{x_{2}-1} dx_{1} = \int_{0}^{1} (x_{1} + \frac{1}{2}) dx_{1} = -\left[\frac{x_{1}^{2}}{2} + \frac{x_{1}}{2} \right]_{0}^{1} = -1$$

On the boundary of the square, the n_2 component is non-zero only on the segments corresponding to $x_2 = 0$ or 1, so that the second integral is:

$$\int_{\partial S} \text{ fg } n_2 \text{ dl} = \int_0^1 (x_1 + 1) (x_1 - 1) \text{ d} x_1 - \int_0^1 x_1^2 \text{ d} x_1 = -1$$

Finally the third integral is written:

$$\int_{0}^{1} \left(\int_{0}^{1} (x_{1} - x_{2}) dx_{2} \right) dx_{1} = \int_{0}^{1} \left[x_{1}x_{2} - \frac{x_{1}^{2}}{2} \right]_{x_{2}=0}^{x_{2}=1} dx_{1} = \int_{0}^{1} (x_{1} - \frac{1}{2}) dx_{1} = \left[\frac{x_{1}^{2}}{2} - \frac{x_{1}}{2} \right]_{0}^{1} = 0$$

Using the numerical values of the three contributions allows verification of the Green theorem for this particular case.

3. A 2-D velocity field is given by the expressions:

$$v_1 = a x_1^2 + 2 b x_1 x_2 + c x_1 x_2$$

$$v_2 = a' x_1^2 + 2 b' x_1 x_2 + c' x_1 x_2$$

depending on the 6 parameters a, b, c, a', b', c'. Determine the relations the parameters must satisfy in order that the velocity field is incompressible. Keeping these relations in mind, show directly that the material flow through the hatched unit triangle pictured below is equal to zero.

- - 1



The partial derivatives are first evaluated by:

$$\frac{Zv_1}{Zx_1} = 2 a x_1 + 2 b x_2 ; \frac{Zv_2}{Zx_2} = 2 b' x_1 + 2 c' x_2$$

They permit us to express the incompressibility equation as:

 $div(\mathbf{v}) = 2(a + b') x_1 + 2(b + c') x_2 = 0$

which can hold in a 2-D domain only if:

a + b' = 0 and b + c' = 0

The hatched triangular domain is denoted by S, with boundary ∂S ; the material flow through ∂S (with outward normal) is evaluated by;

$$\int_{\partial S} \mathbf{v} \cdot \mathbf{n} \, dl = -\int_0^1 v_2 \, dx_1 - \int_0^1 v_1 \, dx_1 + \int_0^1 \frac{(v_1 + v_2)}{\sqrt{2}} \sqrt{2} \, dx_1$$

We see immediately that:

$$\int_0^1 v_2 \, dx_1 = \int_0^1 a' x_1^2 \, dx_1 = \frac{a'}{3}, \text{ and } \int_0^1 v_1 \, dx_1 = \int_0^1 c \, v_2^2 \, dx_1 = \frac{c}{3}$$

The last integral is evaluated on the side with equation $x_2 = 1 - x_1$:

$$\int_0^1 (v_1 + v_2) \, dx_1 = \int_0^1 (ax_1^2 + 2bx_1(1 - x_1) + c(1 - x_1)^2 + a'x_1^2 + 2b'x_1(1 - x_1) + c'(1 - x_1)^2) \, dx_1$$
$$= \frac{1}{3} (a + b + c + a' + b' + c')$$

The three contributions are added to get (a + b + b' + c')/3, which was shown to be equal to zero when the velocity field is incompressible.

4. A porous material with an initial relative density $\rho_r^0 < 1$ is densified by a uniform rate of volume change equal to c. Express the law of relative density ρ_r as a function of time t, the beginning of the process corresponding to t = 0.

We suppose that the initial volume of porous material is V_0 so that the current volume is $V = V_0$ - ct (with c > 0), and the initial mass is m. By definition the density will be:

$$\rho = \frac{m}{V} = \frac{m}{V_0 - ct} = \frac{m}{V_0} \frac{1}{1 - c t/V_0} = \rho_0 \frac{1}{1 - c t/V_0}$$

where ρ_0 is the initial density. Dividing each members of the previous equalities by the maximum density ρ_m (which corresponds to a perfectly dense material), we obtain the relative density:

$$\rho_{\rm r} = \rho_{\rm r0} \frac{1}{1 - c t/V_0} \text{ for } t \le \frac{V_0}{c} (1 - \rho_{\rm r0})$$

as a function of time and the initial relative density ρ_{r0} .

If we consider now that the relative volume rate of change is constant we shall put $\dot{V}/V = -c'$, so that with Eq. 5.42 we can write:

$$\dot{\rho}/\rho = \dot{\rho}_r/\rho_r = - \operatorname{div}(\mathbf{v}) = - \dot{V}/V = c'$$

The last equation is easily integrated with respect to time:

$$ln(\frac{\rho_r}{\rho_{r0}}) = c't$$

and the final expression for relative density change is:

 $\rho_r = \rho_{r0} \exp(c't)$ for $0 \le t \le \frac{-\ln(\rho_{r0})}{c'}$

5. The plane strain upsetting of a rectangular section with height 2h and width 2a is considered (see figure below).



The velocity field at any moment is given by :

$$v_x = + v_0 \frac{x}{h}$$
, $v_y = -v_0 \frac{y}{h}$

where v_0 is the (constant) upsetting velocity.
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Chapter 5

<u>dK</u>

Express the kinematic energy K in the section, and the time derivative \overline{dt} . Calculate the local acceleration of any material point with coordinate (x,y), and the local density of the time derivative of kinematic energy; verify that the integral of this density gives the previous value for \underline{dK} .

SOLUTION:

$$K = \int_{\Omega} \frac{1}{2} \rho \mathbf{v}^2 \, dS = 4 \int_0^n \left(\int_0^a \frac{1}{2} \rho \left(v_0^2 \frac{x^2}{h^2} + v_0^2 \frac{y^2}{h^2} \right) \, dx \right) dy$$

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The kinetic energy K is computed by:

after explicit integration one obtains:
$$K = \frac{2}{3} \rho v_0^2 (\frac{a^3}{h} + ah)$$

$$\frac{\mathrm{dK}}{\mathrm{dt}} = \frac{2}{3} \rho \, \mathrm{v}_0^2 \left(\frac{(3\mathrm{a}^2}{\mathrm{h}} + \mathrm{h}) \, \frac{\mathrm{da}}{\mathrm{dt}} + \left(\frac{-\mathrm{a}^3}{\mathrm{h}^2} + \mathrm{a} \right) \frac{\mathrm{dh}}{\mathrm{dt}} \right)$$

 $\frac{dK}{dt} = \frac{8}{3} \rho v_0^3 \frac{a^3}{h^2}$

The time derivative of the kinetic energy is:

Using the expression of the velocity field, we get: $\frac{da}{dt} = v_x(a) = v_0 \frac{a}{h}$, $\frac{dh}{dt} = v_y(h) = -v_0$

which permits us to rewrite the derivative of the kinematic energy:

Now the acceleration field is calculated by differentiation of the velocity field:

$$\gamma_{x} = \frac{dv_{x}}{dt} = \frac{v_{0}}{h} (v_{x} - \frac{x}{h} \frac{dh}{dt}) = \frac{2 v_{0}^{2}}{h^{2}} x, \quad \gamma_{y} = \frac{dv_{y}}{dt} = -\frac{v_{0}}{h} (v_{y} - \frac{x}{h} \frac{dh}{dt}) = 0$$

and the local rate of change of kinetic energy is: $\rho \mathbf{\gamma} \cdot \mathbf{v} = \frac{2 \rho v_0^3}{h^3} x^2$

The integral of the above function in the rectangular domain is:

$$\int_{\Omega} \rho \boldsymbol{\gamma} \cdot \boldsymbol{v} \, dS = 4 \int_{0}^{h} \left(\int_{0}^{a} \frac{2 \rho v_{0}^{3}}{h^{3}} x^{2} \, dx \right) dy = \frac{8}{3} \rho v_{0}^{3} \frac{a^{3}}{h^{2}}$$
, which is equal to $\frac{dK}{dt}$.

6. With the help of the general variational theory outlined in section 5.5, verify that the functional Π defined by:

$$\Pi(\mathbf{u}) = \frac{1}{2} \int_{\Omega} \left(\lambda \left(\sum_{i} \epsilon_{ii} \right)^2 + 2 \mu \sum_{ij} \epsilon_{ij}^2 \right) d\mathbf{V} - \int_{\partial \Omega^s} \left(\sum_{i} T_i u_i \right) d\mathbf{S}$$

corresponds to the linear elastic problem for a material obeying the Hooke law, with a prescribed stress vector **T** on the boundary $\partial \Omega^{s}$.

SOLUTION:

$$F = \frac{1}{2} \left(\lambda \left(\sum_{i} \epsilon_{ii} \right)^2 + 2 \mu \sum_{i,j} \epsilon_{ij}^2 \right) \text{ and } f = - \prod_i u_i$$

According to Eq. 5.82 we p

$$\frac{\hat{Z}F}{\check{Z}u_{i}} = 0 \quad ; \quad \frac{\hat{Z}F}{\check{Z}\epsilon_{ii}} = \lambda \quad _{i} \quad \epsilon_{ii} + 2 \ \mu \ \epsilon_{ii} = \sigma_{ii} \quad ; \quad \frac{\hat{Z}F}{\check{Z}\epsilon_{ij}} = 2 \ \mu \ \epsilon_{ij} = \sigma_{ij} \quad \text{for } i \neq j$$

According to the Hooke linear elastic theory, the last two terms were identified whith the corresponding stress components, so that Eq. 5.83 is equivalent to the equilibrium equation with $\frac{Z\sigma_{ij}}{Zx_i} = 0$ in Ω

components:

 $\frac{\dot{Z}f}{\dot{Z}u_i} = -T_i, \text{ so that Eq. 5.84 gives:} -T_i + \sigma_{ij} \sigma_$ Now f is differented with respect to u_i: which is the desired boundary condition.

7. A cylindrical sample is considered with length L and radius R, subjected to a prescribed tension force F at its right end, while the left one remains fixed (see figure below).



Compute the energy functional Π (Eq. 5.104) for an elastic material with Lamé coefficients λ and μ . The minimization of Π allows to determine explicitly the unknown parameters U, V, W. Show that the usual simple formulas are obtained when the Young modulus E end the Poisson vcoefficient are used. It is recalled that the following equalities hold:

$$E = \frac{\mu (3\lambda + 2\mu)}{\lambda + \mu} \qquad \nu = \frac{\lambda}{2 (\lambda + \mu)}$$

SOLUTION:

The components of the strain tensor $\mathbf{\varepsilon}$ are derived from the given displacement field:

 $\varepsilon_{11} = U$; $\varepsilon_{22} = V$; $\varepsilon_{33} = W$; $\varepsilon_{ii} = 0$ for $i \neq j$

The prescribed stress in the ox_1 direction on the right hand side of the sample is: $T_1 = \frac{F}{S}$ with $S = \pi R^2$

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Then the functional associated to our problem is written:

$$\Pi(\mathbf{u}) = \frac{1}{2} \int_{\Omega} \left(\lambda \left(U + V + W \right)^2 + 2 \mu \left(U^2 + V^2 + W^2 \right) dV - \int_{\partial \Omega^s} \frac{F}{S} U L dS \right)$$
$$= \frac{1}{2} \pi R^2 L \left(\lambda \left(U + V + W \right)^2 + 2 \mu \left(U^2 + V^2 + W^2 \right) - F U L \right)$$

The minimization of Π with respect to the unknown parameters U, V, W is performed by writing:

$$\frac{\dot{Z}\Pi}{\dot{Z}U} = 0 = \pi R^2 L \left(\lambda \left(U + V + W \right) + 2 \mu U \right) - F I$$
$$\frac{\ddot{Z}\Pi}{\ddot{Z}V} = 0 = \pi R^2 L \left(\lambda \left(U + V + W \right) + 2 \mu V \right)$$
$$\frac{\ddot{Z}\Pi}{\ddot{Z}W} = 0 = \pi R^2 L \left(\lambda \left(U + V + W \right) + 2 \mu W \right)$$

After summing up the above three equations we have:

$$(3 \lambda + 2 \mu) (U + V + W) = \frac{F}{\pi R^2}$$

ain:
$$U = \frac{\lambda + \mu}{\mu} \frac{1}{3\lambda + 2\mu} \frac{F}{\pi R^2} = \frac{1}{E} \frac{F}{\pi R^2}, \quad V = W = \frac{-1}{2\mu} \frac{1}{3\lambda + 2\mu} \frac{F}{\pi R^2} = -\frac{v}{E} \frac{F}{\pi R^2}$$

so that we obta

Find the viscoplastic potential for a material obeying the following constitutive equation: 8. $\mathbf{s} = 2\mathbf{K}(\sqrt{3})^{\mathrm{m-1}} \left(\frac{\dot{\mathbf{\epsilon}}^2}{\mathbf{\epsilon}_0} + \frac{\dot{\mathbf{\epsilon}}^2}{\mathbf{\epsilon}_0} \right)^{\frac{\mathrm{m-1}}{2}} \mathbf{\dot{\mathbf{\epsilon}}}$

where $\dot{\overline{\epsilon}}_0$ is a (small) positive constant. Show that when $\dot{\overline{\epsilon}} << \dot{\overline{\epsilon}}_0$ the material constitutive equation is equivalent to a Newtonian behavior with the general form:

$$\mathbf{s} = 2\eta \, \dot{\mathbf{\epsilon}}$$

and that if $\dot{\overline{\epsilon}} \gg \dot{\overline{\epsilon}}_0$, then it is equivalent to a Norton-Hoff behavior, Eq. 5.110.

SOLUTION:

$$\frac{1}{\epsilon} = \frac{2}{3} \epsilon_{i,j}^2$$

$$d(\dot{\overline{\epsilon}_0}^2 + \dot{\overline{\epsilon}}^2) = \frac{4}{3} \quad \dot{\epsilon}_{ij} \ d\dot{\epsilon}_{ij}$$

from which we deduce immediately:

The definition of the equivalent strain rate is:

$$d\phi = \frac{\dot{Z}\phi}{i,j} \, d\varepsilon_{ij}$$

which is transformed with the help of the constitutive equation and of Eq. 5.106:

$$d\phi = \sum_{i,j} 2 K (\sqrt{3})^{m-1} (\dot{\bar{\epsilon}}_0^2 + \dot{\bar{\epsilon}}^2)^{(m-1)/2} \dot{\bar{\epsilon}}_{ij} d\dot{\bar{\epsilon}}_{ij}$$

On the other hand, for any viscoplastic potential φ we have:

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Using the above result on the differentiation of $(\dot{\overline{\epsilon}}_0^2 + \dot{\overline{\epsilon}}^2)$ we obtain:

$$d\phi = \sum_{i,j} 2 K (\sqrt{3})^{m-1} \left(\dot{\overline{\epsilon}_0}^2 + \dot{\overline{\epsilon}}^2 \right)^{(m-1)/2} \frac{3}{4} d(\dot{\overline{\epsilon}_0}^2 + \dot{\overline{\epsilon}}^2)$$

This equation is easily integrated, giving:

$$\varphi = \frac{K}{m+1} \left(\sqrt{3} \right)^{m+1} \left(\left(\dot{\overline{\epsilon}_0}^2 + \dot{\overline{\epsilon}}^2 \right)^{(m+1)/2} - \dot{\overline{\epsilon}_0}^{m+1} \right)$$

The integration constant was chosen so that $\varphi(0) = 0$. We verify that when $\dot{\overline{\epsilon}}_0 = 0$ the usual viscoplastic potential is found (Eq. 5.110).

B. DEPTH PROBLEMS

9. A potential
$$\varphi(\mathbf{\epsilon})$$
 is convex if for any $\mathbf{\epsilon}_1$ and $\mathbf{\epsilon}_2$ we have:

$$\varphi(\lambda \mathbf{\epsilon}_1 + (1 - \lambda) \mathbf{\epsilon}_2) \check{S} \lambda \varphi(\mathbf{\epsilon}_1) + (1 - \lambda) \varphi(\mathbf{\epsilon}_2) \text{ for } 0 \check{S} \lambda \check{S} 1$$

Apply the above inequality to $\dot{\mathbf{\epsilon}}_1 = 0$ and $\dot{\mathbf{\epsilon}}_2 = \dot{\mathbf{\epsilon}}$ and express the derivative of each side at $\lambda = 0$ to show that:

$$\frac{Z\phi}{\check{Z}\varepsilon}(\varepsilon):\varepsilon \quad \phi(\varepsilon) \quad 0$$

SOLUTION:

The convexity condition is written in the general form:

 $\varphi(\lambda \dot{\boldsymbol{\epsilon}}_1 + (1 - \lambda) \dot{\boldsymbol{\epsilon}}_2) \leq \lambda \varphi(\dot{\boldsymbol{\epsilon}}_1) + (1 - \lambda) \varphi(\dot{\boldsymbol{\epsilon}}_2)$ for $0 \leq \lambda \leq 1$

We put $\dot{\mathbf{e}}_1 = 0$ and $\dot{\mathbf{e}}_2 = \dot{\mathbf{e}}$ in the previous equation so that: $\phi((1 - \lambda) \dot{\boldsymbol{\epsilon}}) \leq (1 - \lambda) \phi(\dot{\boldsymbol{\epsilon}})$ for $0 \leq \lambda \leq 1$

For $\lambda = 0$ both sides of the above equation are equal to $\varphi(\lambda)$: therefore it can be rewritten after substracting $\varphi(\lambda)$ from both sides, dividing by λ and changing the signs:

 $-\frac{\phi((1-\lambda)\,\boldsymbol{\varepsilon})-\phi(\boldsymbol{\varepsilon})}{\lambda} \ge \phi(\boldsymbol{\varepsilon}) \text{ for } 0 \le \lambda \le 1$

When λ tends the left side tends the derivative to zero to that so $\frac{\tilde{Z}\phi(\boldsymbol{\epsilon})}{\check{\boldsymbol{z}}_{\mathbf{c}}} \, \boldsymbol{\epsilon}_{ij} \geq \phi(\boldsymbol{\epsilon}) \ \text{ for } 0 \leq \lambda \leq 1$

which is one of the desired inequalities, the other one resulting from the positiveness of φ which is assumed by hypothesis.

10. Give the proof for the generalized expression of the variational formulation to mechanical problems in section 5.5.

SOLUTION:

We start from the functional of Eq. 5.82 and evaluate a small variation:

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$$\delta I(\mathbf{u}) = \int_{\Omega} \left(\frac{\check{Z}F}{i} \delta u_{i} - \frac{F}{i} \check{Z}\varepsilon_{ij} \delta \varepsilon_{ij} \right) d\mathbf{V} + \int_{\partial\Omega^{s}} \frac{\check{Z}f}{\check{Z}u_{j}} \delta u_{j} d\mathbf{S}$$

Taking into account the symmetry of the strain tensor we verify that:

$$\frac{\tilde{Z}F}{\tilde{Z}\epsilon_{ij}}\,\delta\epsilon_{ij} + \frac{\tilde{Z}F}{\tilde{Z}\epsilon_{ji}}\,\delta\epsilon_{ji} = \frac{\tilde{Z}F}{\tilde{Z}\epsilon_{ij}}\,\delta(\frac{\tilde{Z}u_i}{\tilde{Z}x_j}) + \frac{\tilde{Z}F}{\tilde{Z}\epsilon_{ji}}\,\delta(\frac{\tilde{Z}u_j}{\tilde{Z}x_i})$$

Then the Green theorem is used to write:

$$\int_{\Omega} \frac{\check{Z}F}{\check{Z}\epsilon_{ij}} \frac{\check{Z}(\delta u_i)}{\check{Z}x_j} \, dV = \int_{\partial\Omega} \frac{\check{Z}F}{\check{Z}\epsilon_{ij}} \, \delta u_i \, n_j \, dS - \int_{\Omega} \frac{\check{Z}(\frac{ZT}{\check{Z}\epsilon_{ij}})}{\check{Z}x_j} \, \delta u_i \, dV$$

Recalling that u is prescribed on $\Box \Omega^u$ and has variation δu only on $\Box \Omega^s$, the variation of the functional can be written:

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$$\delta I(\mathbf{u}) = \int_{\Omega} \int_{\Omega} \frac{\check{Z}F}{i} - \frac{\check{Z}(\frac{ZF}{\check{Z}\epsilon_{ij}})}{i} \delta u_i \, dV + \int_{\partial\Omega^{s-i}} \left(\frac{\check{Z}f}{\check{Z}u_i} + \frac{\check{Z}F}{\check{Z}\epsilon_{ij}} n_j\right) \delta u_i \, dS$$

At first we use a displacement field δu which is nill on the boundary: the above equation shows that $\delta I = 0$ only if Eq. 5.83 holds. Then with displacement fields δu which can take any value on the boundary $\Box \Omega^{u}$, we see that the second integral of the above equation must be nill with the consequence that Eq. 5.84 is satisfied.

11. From the variational formulation given by Eq. 5.111, derive the constitutive equation and the equilibrium equation for a Norton-Hoff viscoplastic material (Eq. 5.112). To obtain the appropriate form it is suggested to utilize the following mathematical property of an incompressible vector field v:

if $div(\mathbf{v}) = 0$, then a vector potential $\boldsymbol{\xi}$ exists so that: $\mathbf{v} = \mathbf{curl}(\boldsymbol{\xi})$

First verify that: $\operatorname{curl}(\operatorname{div}(\mathbf{s})) = 0$, where \mathbf{s} is the deviatoric stress tensor. Then, with the help of the previous mathematical result, it is possible to show that $\operatorname{div}(\mathbf{s})$ is the gradient of a scalar field, which is identified with the pressure field p.

SOLUTION:

A variation of the functional is calculated according to:

$$\delta \Phi(\mathbf{v}) = \int_{\Omega} 2K \left(\sqrt{3} \, \dot{\overline{\epsilon}} \right)^{m-1} \dot{\overline{\epsilon}}_{ij} \, \delta \dot{\overline{\epsilon}}_{ij} \, dV - \int_{\partial \Omega^{s-i}} T_i^d \, \delta v_i \, dS$$

Using the constitutive equation (Eq. 5.111) this is equivalent to:

$$\delta \Phi(\mathbf{v}) = \int_{\Omega}^{\mathbf{i}} s_{ij} \, \delta \dot{\varepsilon}_{ij} \, d\mathbf{V} - \int_{\partial \Omega^{s}}^{\mathbf{i}} T_{i}^{d} \, \delta v_{i} \, d\mathbf{S}$$

The Green theorem is used to transform the above equation in the same way as in Problem 10, leading to:

$$\delta \Phi(\mathbf{v}) = \int_{\partial \Omega^{s}} \sup_{i,j} s_{ij} \delta v_i n_j dS - \int_{\Omega} \frac{\partial s_{ij}}{\partial x_j} \delta v_i dV - \int_{\partial \Omega^{s}} T_i^d \delta v_i dS$$

Now δv as well as v are incompressible fields. Then it can be shown mathematically that a vector field $\delta \xi$ does exist so that:

$$\boldsymbol{\delta v} = \mathbf{curl} \left(\boldsymbol{\delta \xi} \right) \text{ with components} \begin{bmatrix} \delta v_1 \\ \delta v_2 \\ \delta v_3 \end{bmatrix} = \begin{bmatrix} \check{Z}/\check{Z}x_2(\delta\xi_3) - \check{Z}/\check{Z}x_3(\delta\xi_2) \\ \check{Z}/\check{Z}x_3(\delta\xi_1) - \check{Z}/\check{Z}x_1(\delta\xi_3) \\ \check{Z}/\check{Z}x_1(\delta\xi_2) - \check{Z}/\check{Z}x_2(\delta\xi_1) \end{bmatrix}$$

At least it is easy to verify that a velocity field δv , derived that way from a vector field $\delta \xi$, is necessarily incompressible.

The only volume integral occuring in the variational form is transformed using the Green theorem to give:

$$\int_{\Omega} -\operatorname{div}(\mathbf{s}) \cdot \operatorname{curl}(\delta \boldsymbol{\xi}) \, \mathrm{dV} = \int_{\Omega} \operatorname{curl}(\operatorname{div}(\mathbf{s}) \cdot \delta \boldsymbol{\xi} \, \mathrm{dV} - \int_{\partial \Omega} \operatorname{div}(\mathbf{s}) \cdot (\mathbf{n} \times \delta \boldsymbol{\xi}) \, \mathrm{dS}$$

If the vector field $\delta \xi$ is chosen with a zero value on the boundary, we obtain:

$$\delta \Phi = \int_{\Omega} \operatorname{curl}(\operatorname{div}(\mathbf{s})).\,\delta \boldsymbol{\xi} \,\mathrm{dV}$$

which can be always null only if $\operatorname{curl}(\operatorname{div}(\mathbf{s})) = 0$. This last condition is mathematically equivalent to the existence of a function, which is denoted by p for convenience, which satisfies $\operatorname{div}(\mathbf{s}) = \operatorname{grad}(p)$.

This relation permits us to substitute div(s) in the initial functional variation:

$$\delta \Phi(\mathbf{v}) = \int_{\partial \Omega^{s-i,j}} s_{ij} \, \delta v_i \, n_j \, dS - \int_{\Omega^{-i}} \frac{\partial p}{\partial x_i} \, \delta v_i \, dV - \int_{\partial \Omega^{s-i}} T_i^d \cdot \delta v_i \, dS$$

Using one more time the Green theorem, we can write:

$$\int_{\Omega} \frac{\partial p}{\partial x_i} \, \delta v_i \, dV = \int_{\partial \Omega^{s-i}} p \, \delta v_i \, n_i \, dS - \int_{\Omega} \frac{\partial \delta v_i}{\partial x_i} \, dV$$

As δv is incompressible the third integral in the above equation is equal to zero, so that:

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$$\delta \Phi(\mathbf{v}) = \int_{\partial \Omega^{s}} s_{ij} \, \delta v_i \, n_j \, dS - \int_{\Omega^{s}} p \, n_i \, \delta v_i \, dV - \int_{\partial \Omega^{s}} T_i^d \, \delta v_i \, dS$$

The velocity variation δv can take any value on the boundary $\partial \Omega^s$, therefore we obtain the equality: $s_{ij} n_j - p n_i = \sigma_{ij} n_j = T_i^d \text{ on } \partial \Omega^s$

Which is the desired boundary condition.

CHAPTER 6 - PROBLEM SOLUTIONS

A. PROFICIENCY PROBLEMS

- 1. Consider a cubic crystal with $c_{11} = 2$, $c_{12} = 1$, and $c_{44} = 1.5$.
 - a. Find c_{12}' , referred to the new coordinate system, if the $\hat{\mathbf{x}}_i'$ axes are rotated 30° counterclockwise about $\hat{\mathbf{x}}_3$. The $\hat{\mathbf{x}}_i$ axes are aligned along the crystal axes.



b. What is the anisotropy parameter for this crystal?

SOLUTION:

[<u>√3</u> 2	$\frac{1}{2}$	0	
[R]=	$-\frac{1}{2}$	<u>√3</u> 2	0	
	0	0	1	

a. According to Chap. 2, Eq. 2.23, the rotation matrix is:

If the fourth rank notation is used for the anisotropic tensor Eq. 6.9 shows that:

$$c'_{12} = c'_{1122} = R_{1m} R_{1n} R_{2o} R_{2p} c_{mnop}$$

Here the summations on m, n, o and p extend from 1 to 2 only as $R_{13} = R_{23} = 0$. Now the only non vanishing terms (with indices 1 or 2) in the c tensor for a cubic material are (see Eq. 6.18):

 $\begin{array}{l} c_{1111} = c_{11}, \, c_{2222} = c_{11}, \, c_{1122} = c_{2211} = c_{12} \\ c_{1212} = c_{2112} = c_{1221} = c_{2121} = c_{44} \end{array}$

The requested component of the c' is then:

$$c'_{1122} = R_{11}^{2} R_{21}^{2} c_{1111} + R_{12}^{2} R_{22}^{2} c_{2222} + R_{11}^{2} R_{22}^{2} c_{1122} + R_{12}^{2} R_{21}^{2} c_{2211} + R_{11} R_{12} R_{21} R_{22} c_{1212} + R_{12} R_{11} R_{21} R_{22} c_{2112} + R_{11} R_{12} R_{22} R_{21} c_{1221} + R_{12} R_{11} R_{21} R_{21} c_{2121} = 2 \frac{3}{4} \frac{1}{4} c_{11} + (\frac{3}{4} \frac{3}{4} + \frac{1}{4} \frac{1}{4}) c_{12} - 4 \frac{3}{4} \frac{1}{4} c_{44} = \frac{3}{8} c_{11} + \frac{5}{8} c_{12} - \frac{3}{4} c_{44} = \frac{3}{8} 2 + \frac{5}{8} 1 - \frac{3}{4} 1.5 = 0.25$$

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- b. The anisotropy ratio is defined in Eq. 6.28: $A = \frac{2 c_{44}}{c_{11} c_{12}} = \frac{2x1.5}{2-1} = 3$
- 2. Repeat Problem 1, b, for $c_{11} = 1$, $c_{12} = 2$, $c_{44} = 0.5$.

SOLUTION:

The same relation as in Problem 1 holds, giving:

$$c'_{12} = \frac{3}{8}c_{11} + \frac{5}{8}c_{12} - \frac{3}{4}c_{44} = \frac{3}{8}1 + \frac{5}{8}2 - \frac{3}{4}0.5 = \frac{5}{4} = 1.25$$

The anisotropy ratio becomes:
$$A = \frac{2 c_{44}}{c_{11} - c_{12}} = \frac{2x0.5}{1-2} = -1$$

3. Use symmetry to reduce the general 6x6 elastic constant matrix to the proper form for a tetragonal cell as shown. (Note: all angles are 90° and a = b, so there is 90° rotational symmetry about $\hat{\mathbf{x}}_2$ and 180° rotational symmetry about \mathbf{x}_3 .)



SOLUTION:

The starting point is the orthotropic material where the anisotropic elasticity constants are given by Eq. 6.17. Additional symmetry is taken into account: the 90° rotation around the $\hat{\mathbf{x}}_2$ vector. The transformation is such that:

				$\sigma_{11} = \sigma_1$	\rightarrow	$\sigma_{33} = \sigma_3$
$\mathbf{\hat{x}}_1$	\rightarrow	$\mathbf{\hat{x}}_3$		$\sigma_{33} = \sigma_3$	\rightarrow	$\sigma_{11} = \sigma_1$
$\mathbf{\hat{x}}_2$	\rightarrow	$\mathbf{\hat{x}}_2$		$\sigma_{23} = \sigma_4$	\rightarrow	- σ ₂₁ = - σ ₆
$\mathbf{\hat{x}}_3$	\rightarrow	$-\mathbf{\hat{x}}_1$	and	$\sigma_{12} = \sigma_6$	\rightarrow	$\sigma_{23} = \sigma_4$

We conclude that the following equalities must hold: $c_{11} = c_{33}$, $c_{12} = c_{23}$, $c_{44} = c_{66}$

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	c ₁₁	c ₁₂	c ₁₃	0	0	0
	c ₁₂	c ₂₂	c ₁₂	0	0	0
	c ₁₃	c ₁₂	c ₁₁	0	0	0
	0	0	0	c44	0	0
	0	0	0	0	c ₅₅	0
The anisotropic coefficients are then:		0	0	0	0	c44

4. a. Is $\psi(x, y) = x^4 + y^4 - 12x^2y^2$ a valid Airy stress function in terms of both equilibrium and compatibility requirements? Show why or why not.

- b. Assuming that the $\psi(x, y)$ in Part a. is correct, find the stress components at the point (1, 2).
- c. Find the strain components for the point (1, 2) assuming plane stress conditions ($\sigma_{31} = 0$). Leave your answers in terms of general Young's modulus (E) and Poisson's ration (V).

SOLUTION:

a. From: $\psi = x^4 + y^4 - 6 x^2 y^2$

we compute: $\frac{\check{Z}^4 \psi}{\check{Z}x^4} = 24$; $\frac{\check{Z}^4 \psi}{\check{Z}y^4} = 24$; $2\frac{\check{Z}^4 \psi}{\check{Z}x^2 \check{Z}y^2} = -48$

which shows that the compatibility condition (Eq. 6.59) is satisfied.

b. The stress components are computed according to:

$$\sigma_{xx} = \frac{\check{Z}^2 \psi}{\check{Z}y^2} = 12 \ y^2 - 12 \ x^2 \ ; \ \sigma_{yy} = \frac{\check{Z}^2 \psi}{\check{Z}x^2} = 12 \ x^2 - 12 \ y^2 \ ; \ \sigma_{xy} = -\frac{\check{Z}^2 \psi}{\check{Z}x \partial y} = 24 \ x \ y$$

The equilibrium equation is automatically satisfied; that can be verified on this peculiar case:

$$\frac{Z\sigma_{xx}}{\check{Z}x} + \frac{Z\sigma_{xy}}{\check{Z}y} = -24 x + 24 x = 0$$
$$\frac{\check{Z}\sigma_{xy}}{\check{Z}x} + \frac{\check{Z}\sigma_{yy}}{\check{Z}y} = 24 y - 24 y = 0$$

c. The corresponding strain components are as follows

$$\varepsilon_{xx} = \frac{1+\nu}{E} \sigma_{xx} - \frac{1}{E} (\sigma_{xx} + \sigma_{yy}) = 12 \frac{1+\nu}{E} (y^2 - x^2)$$

$$\varepsilon_{yy} = \frac{1+\nu}{E} \sigma_{yy} - \frac{1}{E} (\sigma_{xx} + \sigma_{yy}) = 12 \frac{1+\nu}{E} (x^2 - y^2)$$

$$\varepsilon_{xy} = \frac{1+\nu}{E} \sigma_{xy} = 24 \frac{1+\nu}{E} xy$$

- 5. Given: $\psi = 3xy^5 + 3x^5y 10x^3y^3$
 - a. Verify that ψ meets the condition for a proper Airy stress Function. Show your work.

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- b. Find the stresses as functions of x and y that correspond to ψ .
- c. Check to see if the stresses in Part b. satisfy equilibrium. Show your work.
- d. With a little manipulation using the stresses in Part b and Hooke's Law for plane stress, obtain the strain functions shown below. Verify that these strain functions satisfy the compatibility conditions for strains. Show your work.

SOLUTION:

a. The Airy function has the form: $\Psi = 3 \times y^5 5 + 3 \times y^5 - 10 \times y^3 y^3$

We see that the partial derivatives in Eq. 6.59 are:

$$\frac{\ddot{Z}^{4}\psi}{\check{Z}x^{4}} = 360 \text{ x y }; \ \frac{\ddot{Z}^{4}\psi}{\check{Z}y^{4}} = 360 \text{ x y }; \ \frac{\ddot{Z}^{4}\psi}{\check{Z}x^{2} \partial y^{2}} = -360 \text{ x y }$$

so that Eq. 6.59 which expresses the compatibility condition is satisfied.

$$\sigma_{xx} = \frac{\check{Z}^2 \psi}{\check{Z}y^2} = 60 (x \ y^3 - x^3 \ y)$$

$$\sigma_{yy} = \frac{\check{Z}^2 \psi}{\check{Z}x^2} = 60(x^2 \ y - 12 \ x \ y^3)$$

$$\sigma_{xy} = -\frac{\check{Z}^2 \psi}{\check{Z}x \ \partial y} = -15 (x^4 + y^4) + 90 \ x^2 \ y^2$$

- b. We deduce the stress components according to:
- c. The equilibrium equation is automatically verified by stress components deduced from an Airy function, on our example we have:

$$\frac{Z\sigma_{xx}}{Zx} + \frac{Z\sigma_{xy}}{Zy} = 60 \text{ y}^3 - 180 \text{ x}^2 \text{ y} + (-60 \text{ y}^3 + 180 \text{ x}^2 \text{ y}) = 0$$
$$\frac{Z\sigma_{xy}}{Zx} + \frac{Z\sigma_{yy}}{Zy} = -60 \text{ y}^3 + 180 \text{ x}^2 \text{ y} + 60 \text{ y}^3 - 180 \text{ x}^2 \text{ y} = 0$$

$$\epsilon_{xx} = 4 b (xy^3 - x^3y)$$

$$\epsilon_{yy} = 4 b (x^3y - xy^3)$$

$$\epsilon_{xy} = b (-x^4 - y^4 + 6x^2y^2)$$

where: $b \equiv 15 \frac{(1 + v)}{E}$

d. The components of the strain tensor are calculated in the following way:

$$\epsilon_{xx} = \frac{1+\nu}{E} \sigma_{xx} - \frac{1}{E} (\sigma_{xx} + \sigma_{yy}) = 60 \frac{1+\nu}{E} (x y^3 - x^3 y)$$

$$\epsilon_{yy} = \frac{1+\nu}{E} \sigma_{yy} - \frac{1}{E} (\sigma_{xx} + \sigma_{yy}) = 60 \frac{1+\nu}{E} (x^3 y - x y^3)$$

$$\epsilon_{xy} = \frac{1+\nu}{E} \sigma_{xy} = \frac{1+\nu}{E} (-15 x^4 - 15 y^4 + 90 x^2 y^2)$$

When we put $b = 15 \frac{1 + v}{E}$, the desired form is obtained.

The compatibility condition is written here:

$$\frac{\check{Z}^2 \varepsilon_{xx}}{\check{Z}y^2} + \frac{\check{Z}^2 \varepsilon_{yy}}{\check{Z}x^2} - 2\frac{\check{Z}^2 \varepsilon_{xy}}{\check{Z}x \,\partial y} = 24 \text{ bxy} + 24 \text{ bxy} - 2 (24 \text{ bxy}) = 0$$

6. Verify that Eqs. 6.38-6.42 follow from the traditional definitions of elastic constants.

SOLUTION:

We consider a uniaxial tensile test in the **x** direction. Hooke's law in terms of Lamé coefficients (Eqs.6.31a-f) reduces to:

$$\sigma_{xx} = \lambda (\varepsilon_{xx} + 2 \varepsilon_{yy}) + 2 \mu \varepsilon_{xx}$$

$$\sigma_{yy} = \sigma_{zz} = \lambda (\varepsilon_{xx} + 2 \varepsilon_{yy}) + 2 \mu \varepsilon_{yy} = 0$$

These equations can be compared to Eq. 6.1 - 5 for the same case:

$$\varepsilon_{xx} = \frac{1}{E} \sigma_{xx}$$
; $\varepsilon_{yy} = \frac{-v}{E} \sigma_{xx}$ or $\varepsilon_{yy} = -v \varepsilon_{xx}$

We obtain the equalities:

$$\sigma_{xx} = E \varepsilon_{xx} = (\lambda(1 - 2 \nu) + 2 \mu) \varepsilon_{xx}$$

$$\sigma_{yy} = (\lambda(1 - 2 \nu) - 2 \mu \nu) \varepsilon_{xx} = 0$$

From which we deduce by subtracting the second one to the first one: $\mu = \frac{E}{2(1+\nu)}$

Then μ is immediately deduced (provided $1 - 2\nu \neq 0$): $\lambda = \frac{E\nu}{2(1 + \nu)(1 - 2\nu)}$

From the Hooke's equation with the Lamé coefficients the equality $G = \mu$ is obvious. The bulk modulus B is defined by: $B = \frac{\frac{1}{3}(\sigma_{11} + \sigma_{22} + \sigma_{33})}{\epsilon_{11} + \epsilon_{22} + \epsilon_{33}} = \frac{1}{K}$

The easiest way to establish the relation is to consider a spherical strain tensor so that $\varepsilon_{xx} = \varepsilon_{yy} = \varepsilon_{zz} = \varepsilon$. According to Hooke's law, the stress state is $\sigma_{xx} = \sigma_{yy} = \sigma_{zz} = (3 \ \lambda + 2 \ \mu) \varepsilon$, and the bulk modulus is: $B = \lambda + \frac{2}{3} \mu$

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The other form of Hooke's equation is here:

$$B = \frac{\sigma_{xx}}{3 \frac{1-2\nu}{E} \sigma_{xx}} = \frac{E}{3 (1-2\nu)}$$

 $\varepsilon_{xx} = \frac{1}{E} \sigma_{xx} - \frac{v}{E} (\sigma_{yy} + \sigma_{zz}) = \frac{1-2v}{E} \sigma_{xx}$

so that the bulk modulus becomes:

Finally the relation between λ , μ and E, ν can be written: 2 (1 + ν) μ = E (1 + ν) (1 - 2 ν) λ = E ν or (1 - 2 ν) $\frac{\lambda}{\mu}$ = ν

 $v = \frac{\lambda}{2(\lambda + \mu)}$ and $E = \frac{(3\lambda + 2\mu)\mu}{\lambda + \mu}$

from which we deduce easily:

7. Solve Eqs. 6.31 for strains and verify that the result is identical to Eqs. 6.1-5.

SOLUTION:

First we see from Eqs. 6.31d-f and Eq. 6.38: for $i \neq j$ $\varepsilon_{ij} = \frac{1}{2\mu} \sigma_{ij} = \frac{1+\nu}{E} \sigma_{ij}$

$$\int_{k=1}^{3} \sigma_{kk} = (3 \ \lambda + 2 \ \mu) \int_{k=1}^{3} \varepsilon_{kk} \text{ or } \int_{k=1}^{3} \varepsilon_{kk} = \frac{1}{3 \ \lambda + 2 \ \mu} \int_{k=1}^{3} \sigma_{kk}$$

Then we add Eqs. 6.31a-c:

$$\varepsilon_{ii} = 2 \mu \sigma_{ii} - \frac{\lambda}{2 \mu (3 \lambda + 2 \mu)} \int_{k=1}^{3} \sigma_{kk}$$

Eq. 6.31a-c can be rewritten:

Using Eqs. 6.38-39 and 6.42, we obtain: $\frac{\lambda}{2 \mu (3 \lambda + 2 \mu)} = \frac{\nu}{E}$, so that the last equation is transformed into: $\epsilon_{ii} = \frac{1 + \nu}{E} \sigma_{ii} - \frac{\nu}{E} \sum_{k=1}^{3} \sigma_{kk}$, which is the desired result.

B. DEPTH PROBLEMS

8. Discuss any material restrictions which apply to the compatibility conditions Eqs. 6.52 and 6.55. Consider anisotropy, elasticity vs. plasticity or other constitutive equations, possible presence of body forces, and the possibility of voids and cracks developing during deformation.

SOLUTION:

Eq. 6.52 does not use any additional physical assumption regarding the material. It is only based on the continuity of the third order derivatives of the displacement field. In the presence of cracks or a non differentiable boundary, this hypothesis must be analyzed carefully as the

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displacement field may be discontinuous, thus violating the idea of a continuum of material upon which compatibility is based.

On the other hand, Eq. 6.55 is established from Eq. 6.52 and the constitutive equation, so that it can be valid only for isotropic linear elasticity (with or without body forces). This equation must not be used for anisotropic elasticity, plasticity, elastoplasticity, etc.... An equivalent form can be derived for a given relationship between stress and strain for other cases.

9. *a.* Show that the elastic work done during small straining is given by $w = \frac{1}{2} c_{jkl} \epsilon_{ij} \epsilon_{kl}.$

- b. Write the elastic work in terms of stresses alone.
- c. Write results for parts a and b for the isotropic case.

SOLUTION:

a. We suppose for convenience that the strain is applied with a so called radial loading, i. e. we suppose that loading is performed from 0 to t, and that for any time τ , with $0 \le \tau \le t$, the strain tensor is: $\mathbf{\epsilon}^{\tau} = \frac{\tau}{t} \mathbf{\epsilon}$ and the corresponding stress tensor: $\mathbf{\sigma}^{\tau} = \mathbf{c} : \mathbf{\epsilon}^{\tau} = \frac{\tau}{t} \mathbf{c} : \mathbf{\epsilon}$. During the time increment $d\tau$ the strain increment is: $d\mathbf{\epsilon}^{\tau} = \frac{d\tau}{t} \mathbf{\epsilon}$, so that the increment of work is deduced with the help of Eq. 4.54: $dw = \mathbf{\sigma}^{\tau} : d\mathbf{\epsilon}^{\tau} = \frac{\tau}{t} (\mathbf{c} : \mathbf{\epsilon}) : \mathbf{\epsilon} \frac{d\tau}{t}$

$$\mathbf{w} = \int_0^t \frac{\mathbf{\tau}}{t} (\mathbf{c} : \boldsymbol{\varepsilon}) : \boldsymbol{\varepsilon} \frac{d\mathbf{\tau}}{t} = \frac{1}{2} (\mathbf{c} : \boldsymbol{\varepsilon}) : \boldsymbol{\varepsilon} = \frac{1}{2} c_{ijkl} \varepsilon_{ij} \varepsilon_{kl}$$

The total work is obtained by time integration:

The assumption of radial loading is not required since, by definition, the elastic response is conservative, i. e. no work is done in a closed loading cycle returning to the same loading state.

- b. The compliance tensor s is introduced (see Exercise 6.1) which permits us to write: $\mathbf{\varepsilon} = \mathbf{s} : \mathbf{\sigma}$ and $\mathbf{w} = \frac{1}{2} (\mathbf{s} : \mathbf{\sigma}) : \mathbf{\sigma} = \int_{ijkl} \frac{1}{2} \mathbf{s}_{ijkl} \ \sigma_{ij} \ \sigma_{kl}$ the work in the form:
- c. The isotropic expressions are derived from Eqs. 6.31 and 6.1 4:

$$\mathbf{w} = \frac{\lambda}{2} \begin{pmatrix} & \mathbf{\varepsilon}_{ii} \end{pmatrix}^2 + \mu & \mathbf{\varepsilon}_{ij}^2 = \frac{-\mathbf{v}}{2\mathbf{E}} \begin{pmatrix} & \mathbf{\sigma}_{ii} \end{pmatrix}^2 + \frac{1+\mathbf{v}}{2\mathbf{E}} & \mathbf{\sigma}_{ij}^2 \end{pmatrix}$$

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b. Find the equilibrium equation in indicial form in terms of strains alone.

SOLUTION:

$$\sigma_{ij} = \lambda \, \epsilon_{kk} \, \delta_{ij} + 2 \, \mu \, \epsilon_{ij} = 2 \, \mu \, (\epsilon_{ij} + \frac{\lambda}{2 \, \mu} \, \epsilon_{kk} \, \delta_{ij})$$

a. Hooke's law is written:

with the help of Eq. 6.39 it is easy to obtain the desired result: $\sigma_{ij} = 2 \mu (\epsilon_{ij} + \frac{\nu}{1 - 2\nu} \epsilon_{kk} \delta_{ij})$

$$\operatorname{div}(\boldsymbol{\sigma})_{i} = \frac{\tilde{Z}\sigma_{ij}}{\tilde{Z}x_{j}} = 2 \ \mu \ (\frac{\tilde{Z}\varepsilon_{ij}}{\tilde{Z}x_{j}} + \frac{\nu}{1 - 2\nu} \frac{\tilde{Z}\varepsilon_{kk}}{\tilde{Z}x_{j}} \delta_{ij}) = 0$$

- b. The equilibrium equation is expressed as:
- 11. Consider a plane-strain isotropic linear elastic problem defined in a given domain. Assume the displacement vector field takes the form:

=	$\mu_1 = a x_1^2 + 2b x_1 x_2 + c x_2^2$
u –	$\mu_2 = a' x_1^2 + 2b' x_1x_2 + c' x_2^2$

- a. Write the strain tensor as a function of x_1 and x_2 .
- b. Compute the stress tensor as a function of x_1 and x_2 with the isotropic Hooke law.
- *c.* Write the stress equilibrium equation assuming no body forces, and determine the relation between the coefficients *a*, *b*, *c*, *a'*, *b'*, and *c'* that we must impose.
- d. Calculate the stress vector on each side of a square (see figure below).



Verify that the external forces on the square are in equilibrium when the condition defined in Part c is fulfilled.

SOLUTION:

a. The derivatives of the displacement components are $\frac{\check{Z}u_1}{\check{Z}x_1} = 2 a x_1 + 2 b x_2, \quad \frac{\check{Z}u_1}{\check{Z}x_2} = 2 b x_1 + 2 c x_2$ $\frac{\check{Z}u_2}{\check{Z}x_1} = 2 a' x_1 + 2 b' x_2, \quad \frac{\check{Z}u_2}{\check{Z}x_2} = 2 b' x_1 + 2 c' x_2$

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The strain rate tensor is immediately obtained: $[\varepsilon] = \begin{bmatrix} 2 \ a \ x_1 + 2 \ b \ x_2 & (b+a') \ x_1 + (c+b')x_2 \\ (b+a') \ x_1 + (c+b')x_2 & 2 \ b' \ x_1 + 2 \ c' \ x_2 \end{bmatrix}$

$$\sigma_{11} = 2\lambda ((a+b') x_1 + (b+c') x_2) + 4 \mu (a x_1 + b x_2)$$

$$\sigma_{12} = 2 \mu ((b+a') x_1 + (c+b') x_2)$$

-

b. The components of stress tensor are now: $\sigma_{22} = 2\lambda ((a+b') x_1 + (b+c') x_2) + 4 \mu (b' x_1 + c' x_2)$

$$\frac{\check{Z}\sigma_{11}}{\check{Z}x_1} + \frac{\check{Z}\sigma_{12}}{\check{Z}x_2} = 2 \lambda (a+b') + 4 \mu a + 2 \mu (c+b') = 0$$

c. The equilibrium equation is written:
$$\frac{\check{Z}\sigma_{12}}{\check{Z}x_1} + \frac{\check{Z}\sigma_{22}}{\check{Z}x_2} = 2 \mu (b+a') + 2 \lambda (b+c') + 4 \mu c' = 0$$

d. The stress vector on the upper side and its integral on this side are:

$$\begin{bmatrix} \sigma_{12} \\ \sigma_{22} \end{bmatrix} = \begin{bmatrix} 2 \mu ((b+a') x_1 + (c+b')) \\ 2\lambda ((a+b') x_1 + (b+c')) + 4 \mu (b' x_1 + c') \end{bmatrix}, \int_{-1}^{1} \begin{bmatrix} \sigma_{12} \\ \sigma_{22} \end{bmatrix} dx_1 = \begin{bmatrix} 4 \mu (c+b') \\ 4 \lambda (b+c') + 8 \mu c' \end{bmatrix}$$

On the lower side:

$$\begin{bmatrix} -\sigma_{12} \\ -\sigma_{22} \end{bmatrix} = \begin{bmatrix} -2\mu((b+a')x_1 - (c+b')) \\ -2\lambda((a+b')x_1 - (b+c')) - 4\mu(b'x_1 - c') \end{bmatrix}_{2} \int_{-1}^{1} \begin{bmatrix} -\sigma_{12} \\ -\sigma_{22} \end{bmatrix} dx_1 = \begin{bmatrix} 4\mu(c+b') \\ 4\lambda(b+c') + 8\mu c' \end{bmatrix}$$

On the right hand side:

$$\begin{bmatrix} \sigma_{11} \\ \sigma_{12} \end{bmatrix} = \begin{bmatrix} 2\lambda \left((a+b') + (b+c') x_2 \right) + 4\mu (a+b x_2) \\ 2\mu \left((b+a') + (c+b') x_2 \right) \end{bmatrix}, \int_{-1}^{1} \begin{bmatrix} \sigma_{11} \\ \sigma_{12} \end{bmatrix} dx_2 = \begin{bmatrix} 4\lambda (a+b') + 8\mu a \\ 4\mu (b+a') \end{bmatrix}$$

On the left hand side:

$$\begin{bmatrix} -\sigma_{11} \\ -\sigma_{12} \end{bmatrix} = \begin{bmatrix} -2\lambda(-(a+b') + (b+c')x_2) - 4\mu(a+bx_2) \\ -2\mu((b+a') - (c+b')x_2) \end{bmatrix}, \int_{-1}^{1} \begin{bmatrix} -\sigma_{11} \\ -\sigma_{12} \end{bmatrix} dx_2 = \begin{bmatrix} 4\lambda(a+b') + 8\mu a \\ 4\mu(b+a') \end{bmatrix}$$

When all these contribution are added and the result equated to zero we obtain:

$$4 \mu (c+b') + 4 \lambda (a+b') + 8 \mu a = 0$$

$$4 \lambda (b+c') + 8 \mu c' + 4 \mu (b+a') + 0$$
, which are equivalent to the equations we found in c

12. Consider 3 elastic bars of equal length which are pinned at the ends as shown and to which a vertical force F is applied.



Each bar has a length l, a section area s, a Young's modulus E. We assume the links between the bars do not permit any torque, the bars remain straight and the contact with the horizontal plane is frictionless. Compute the variation of length of each bar, with the hypothesis of small displacements. Consider the case where the variation of length is no longer negligible as compared to the initial geometry.

SOLUTION:

Qualitatively we can see that a compression force F_1 will occur in the two rods AB and AC, while the horizontal rod BC will be subjected to a tension force F_2 .



The equilibrium of vertical forces at point A gives: $F = 2 F_1 \sin(60^\circ)$ or $F_1 = F/\sqrt{3}$. If the length variation Δl_1 of AB or AC is introduced, we have also: $F_1 = F/\sqrt{3} = E a \frac{\Delta l_1}{l}$ or $\Delta l_1 = \frac{F l}{E a \sqrt{3}}$

The horizontal force exerted in B on rod BC by rod AB is: $F_2 = F_1 \cos(60^\circ) = \frac{F}{2\sqrt{3}}$ and the length variation Δl_2 of BC is computed by: $F_2 = \frac{F}{2\sqrt{3}} = E a \frac{\Delta l_2}{l} \text{ or } \Delta l_2 = \frac{Fl}{2\sqrt{3}} E a$

When the displacements are no longer neglected, we suppose that the new lengths of the rods are:

$$AB = AC \rightarrow l_1 = l + \Delta l_1; BC \rightarrow l_2 = l + \Delta l_2$$

$$F_{1} = \frac{F}{2 \sin \alpha} = \frac{F}{2} \frac{1 + \Delta l_{1}}{\sqrt{(1 + \Delta l_{1})^{2} - \frac{1}{4}(1 + \Delta l_{2})^{2}}} = E a \frac{\Delta l_{1}}{l}$$

The same approach gives:

$$F_{2} = F_{1} \cos \alpha = \frac{F}{2} \frac{\cos \alpha}{\sin \alpha} = \frac{F}{4} \frac{1 + \Delta l_{2}}{\sqrt{(1 + \Delta l_{1})^{2} - \frac{1}{4}(1 + \Delta l_{2})^{2}}} = E a \frac{\Delta l_{2}}{l}$$

and for rod BC:

We obtain a non linear system of two equations with the unknown Δl_1 and Δl_2 , which can be solved iteratively with the Newton-Raphson method. Here we assumed Hooke's law and suppose the sections of the rods did not change: the approach can then be still improved be considering a nonlinear elasticity law and taking into account the area changes.

13. Consider a plane-strain compression test of a sample with height h, imposing a displacement u at the top as shown in the figure, assuming no friction on the plates.



Write the displacement field using a linear expression and taking into account the boundary conditions and the symmetry of the problem: an unknown parameter α should be introduced in the x_1 component. Calculate the strain tensor, the stress tensor with the isotropic Hooke law, and the elastic energy. Show that the solution of the problem can be obtained by minimizing the elastic energy with respect to the unknown parameter α . Verify that the stress on the vertical sides are equal to zero.

SOLUTION:

$$[\mathbf{u}] = \begin{bmatrix} u_1 \\ u_2 \end{bmatrix} = \begin{bmatrix} \alpha x_1 \\ -u \frac{x_2}{h} \end{bmatrix}; \quad [\varepsilon] = \begin{bmatrix} \alpha & 0 \\ 0 & -\frac{u}{h} \end{bmatrix}$$

ields are:

The displacement and strain fields are:

and the stress tensor is deduced immediately with the help of the Hooke law:

$$[\sigma] = \begin{bmatrix} \lambda (\alpha - \frac{\mathbf{u}}{\mathbf{h}}) + 2\mu \alpha & 0\\ 0 & \lambda (\alpha - \frac{\mathbf{u}}{\mathbf{h}}) - 2\mu \frac{\mathbf{u}}{\mathbf{h}} \end{bmatrix}$$

The elastic density of energy w is homogeneous in the 2-D domain so that it is easily integrated to give: $W = \frac{1}{2} \left(\lambda \left(\alpha - \frac{u}{h} \right)^2 + 2\mu \left(\alpha^2 - \left(\frac{u}{h} \right)^2 \right) \right) 2 a h$

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where 2 a is the width of the sample. Minimization of the elastic energy yields: $\frac{ZW}{Z\alpha} = 0 = \left(\lambda \left(\alpha - \frac{u}{h}\right) + 2\mu \alpha\right) 2 a h$

The optimum parameter is: $\alpha = \frac{\lambda}{\lambda + 2 \mu} \frac{u}{h}$ and the corresponding stress tensor is written:

$$[\sigma] = \begin{bmatrix} 0 & 0 \\ 0 & -4\mu \frac{\lambda + \mu}{\lambda + 2\mu} \lambda \frac{\mu}{h} \end{bmatrix}$$

We see that it is the exact solution of our problem.

14. Repeat Problem 13 for a cubic crystal where \mathbf{x}_1 and \mathbf{x}_2 are oriented along four-fold symmetry axes.

SOLUTION:

The displacement and strain fields are the same as those in problem 13, the non zero components of the stress tensor become:

$$\sigma_{11} = c_{11} \epsilon_{11} + c_{12} \epsilon_{22}$$

$$\sigma_{22} = c_{12} \epsilon_{11} + c_{11} \epsilon_{22}$$

The elastic energy is derived according to:

W =
$$\frac{1}{2} (c_{11} \epsilon_{11}^2 + 2 c_{12} \epsilon_{22} \epsilon_{11} + c_{11} \epsilon_{22}^2) 2 a h$$

The minimization of the elastic energy is achieved by putting:

$$\frac{\dot{Z}W}{\check{Z}\alpha} = (c_{11} \alpha + c_{12} (-\frac{u}{h})) 2 a h \implies \alpha = \frac{c_{12}}{c_{11}} \frac{u}{h}$$

The final solution in stress is:

$$\sigma_{11} = 0$$
, $\sigma_{22} = -\frac{c_{11}^2 - c_{12}^2}{c_{11}} \frac{u}{h}$

15. Repeat Problem 13 with a 3-D sample and an orthotropic material.

SOLUTION:

The approach is similar to that of problems 13 and 14. We have:

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$$\begin{bmatrix} u_1 \\ u_2 \\ u_3 \end{bmatrix} = \begin{bmatrix} \alpha & x_1 \\ \beta & x_2 \\ -\frac{u}{h} & x_3 \end{bmatrix} \quad [\varepsilon] = \begin{bmatrix} \alpha & 0 & 0 \\ 0 & \beta & 0 \\ 0 & 0 & -\frac{u}{h} \end{bmatrix}$$

The non-zero components of the stress tensor are calculated with Eq. 6.17:

$$\sigma_{11} = c_{11} \alpha + c_{12} \beta + c_{13} \left(-\frac{u}{h}\right)$$

$$\sigma_{22} = c_{12} \alpha + c_{22} \beta + c_{23} \left(-\frac{u}{h}\right)$$

$$\sigma_{33} = c_{13} \alpha + c_{23} \beta + c_{33} \left(-\frac{u}{h}\right)$$

and the elastic energy takes the form:

$$W = \frac{1}{2} 4abh \left(c_{11} \alpha^2 + c_{22} \beta^2 + c_{33} (-\frac{u}{h})^2 + 2 c_{12} \alpha \beta + 2 c_{13} \alpha (-\frac{u}{h}) + 2 c_{23} \beta (-\frac{u}{h}) \right)$$

if 2a is the length and 2b the width of the sample. Minimization with respect to the two unknown parameters gives the linear system of two equations:

$$\begin{split} \frac{\check{Z}W}{\check{Z}\alpha} &= 0 = 4abh\left(c_{11} \ \alpha + c_{12} \ \beta + c_{13}(-\frac{u}{h})\right) \Rightarrow \ \sigma_{11} = 0\\ \frac{\check{Z}W}{\check{Z}\beta} &= 0 = 4abh\left(c_{12} \ \alpha + c_{22} \ \beta + c_{33}(-\frac{u}{h})\right) \Rightarrow \ \sigma_{22} = 0 \end{split}$$

$$\alpha = \frac{c_{22} c_{13} - c_{12} c_{23}}{c_{11} c_{22} - c_{12}^2} \frac{u}{h}$$
$$\beta = \frac{c_{11} c_{23} - c_{12} c_{13}}{c_{11} c_{22} - c_{12}^2} \frac{u}{h}$$

The solution of which is:

16. Show that the <u>only</u> incompressible isotropic elastic medium is a liquid by computing the shear modulus, μ .

SOLUTION:

An isotropic linear elastic medium which obeys Hooke's law is incompressible if (see Eq. 6.1 - 5): $0 = \varepsilon_{11} + \varepsilon_{22} + \varepsilon_{33} = \frac{1 - 2\nu}{E} (\sigma_{11} + \sigma_{22} + \sigma_{33}), \text{ which holds when } \nu = 0.5.$ According to Eq. 6.41 we can also write: $\frac{1}{2} = \frac{\lambda}{2(\lambda + \mu)}, \text{ that is } \mu = 0 \text{ for any finite value of } \lambda, \text{ so}$

According to Eq. 6.41 we can also write: $^{2} 2(\lambda + \mu)$, that is $\mu = 0$ for any finite value of λ , so that $\sigma_{ij} = 0$ for any $i \neq j$. Moreover the diagonal terms of the stress tensor do not depend on the strain tensor so that by isotropy we have: $\sigma_{11} = \sigma_{22} = \sigma_{33} = -p$. That corresponds to an inviscid fluid.

17. The Airy Stress Function for a screw dislocation in an isotropic crystal is given by

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$$\psi = \frac{\mu by}{4\pi (1-v)} \ln (x^2 + y^2)$$
, where b = Burger's vector, a small constant.

- a. Find the stresses and strains for the screw dislocation.
- b. Is compatibility satisfied everywhere? Why or why not?
- c. Do plane strain or plane stress condition hold?

SOLUTION:

a. We first put $k = \frac{\mu b}{4\pi (1 - \nu)}$ and $r^2 = x^2 + y^2$ in order to simplify the expressions. Eq. 6.46 gives the stress components:

$$\sigma_{xx} = \frac{\check{Z}^2 \psi}{\check{Z}y^2} = 2 \text{ k y } \frac{3 \text{ } x^2 + y^2}{r^4}, \ \sigma_{yy} = \frac{\check{Z}^2 \psi}{\check{Z}x^2} = 2 \text{ k y } \frac{y^2 - x^2}{r^4}, \ \sigma_{xy} = -\frac{\check{Z}^2 \psi}{\check{Z}x \partial y} = 2 \text{ k x } \frac{y^2 - x^2}{r^4}$$

According to Eq. 6.1 - 5 the strain components are:

$$\varepsilon_{xx} = 2 k \frac{y}{r^4} \left(\frac{1}{E} (3 x^2 + y^2) - \frac{v}{E} (y^2 - x^2) \right), \quad \varepsilon_{yy} = 2 k \frac{y}{r^4} \left(\frac{-v}{E} (3 x^2 + y^2) + \frac{1}{E} (y^2 - x^2) \right)$$

$$\varepsilon_{zz} = \frac{-v}{E} 4 k \frac{y}{r^2}, \qquad \varepsilon_{xy} = \frac{1 + v}{E} 2 k x \frac{y^2 - x^2}{r^4}$$

 $\varphi = \frac{\check{Z}^2 \psi}{\check{Z}x^2} + \frac{\check{Z}^2 \psi}{\check{Z}y^2} = \frac{4 \text{ k y}}{r^2}$ b. We put:

tte:
$$\frac{\check{Z}^2 \varphi}{\check{Z}x^2} = \frac{-8 \text{ k y } (y^2 - 3 \text{ x}^2)}{r^2}$$
 and $\frac{\check{Z}^2 \varphi}{\check{Z}y^2} = \frac{8 \text{ k y } (y^2 - 3 \text{ x}^2)}{r^2}$

we have to compute:

so that the compatibility condition:
$$\frac{\check{Z}^2 \varphi}{\check{Z} x^2} + \frac{\check{Z}^2 \varphi}{\check{Z} y^2} = 0$$
 is fulfilled everywhere except at the origin.

- c. The mechanical problem is obviously plane stress but not plane strain.
- 18. The elastic fields of a screw dislocation are most simply derived by considering the displacements in polar coordinates:

 $u_z(r, \theta) = \frac{b\theta}{2\pi} = \frac{b}{2\pi} \tan^{-1} \frac{y}{x}$, where b = the Burger's vector, a small constant.

- a. Find the stresses and strains for the screw dislocation.
- b. Is compatibility satisfied everywhere? Why or why not?

c. Do plane strain or plane stress conditions hold?

SOLUTION:

a. From the given displacement field we compute the strain and stress tensors in the usual way:

$$\begin{bmatrix} u_{x} = 0 \\ u_{y} = 0 \\ u_{z} = \frac{b}{2\pi} \tan^{-1} \frac{y}{x} \end{bmatrix} \rightarrow [\varepsilon] = \begin{bmatrix} 0 & 0 & -\frac{b'}{2} \frac{y}{r^{2}} \\ 0 & 0 & \frac{b'}{2} \frac{x}{r^{2}} \\ -\frac{b'}{2} \frac{y}{r^{2}} & \frac{b'}{2} \frac{x}{r^{2}} & 0 \end{bmatrix} \rightarrow [\sigma] = \begin{bmatrix} 0 & 0 & -\mu b' \frac{y}{r^{2}} \\ 0 & 0 & \mu b' \frac{x}{r^{2}} \\ -\mu b' \frac{y}{r^{2}} & \mu b' \frac{x}{r^{2}} & 0 \end{bmatrix}$$

where we put: $r^2 = x^2 + y^2$ and $b' = \frac{b}{2\pi}$

b. Compatibility is necessarily satisfied everywhere (except at the origin) as the strain tensor is obtained by differentiation of a displacement field.

c. By examining the strain and stress tensors we conclude that we have both plane strain and stress tensor fields (with respect to plane Oxy).

CHAPTER 7 - PROBLEM SOLUTIONS

A. PROFICIENCY PROBLEMS

1. What is the meaning of stress states lying outside of yield surface?

SOLUTION:

They are states accessible only by changing the material properties - by strain hardening or heat treatment perhaps. In either case, the yield surface must change in order to achieve these stresses.

2. A researcher has found a way to measure the yield surface of sheet metal rapidly and automatically. He inserts a sheet into a biaxial testing machine and loads along proportional paths until, while he measures strains in the two directions, he obtains the 0.2% offset strength. Then he unloads, chooses a slightly different ratio and does the same thing, until he has generated many yield points.

Criticize this procedure.

SOLUTION:

The major criticism is that the material properties are different after the first and each subsequent test because the approximately 0.002 plastic strains accumulate and the material strain hardens. Therefore, the various yield stresses measured are not for a single material and do not represent a yield surface at an instant. A secondary criticism is the use of the 0.002 offset strength as the yield strength. In practice, this is not a very practical problem, but the yield strength measured in this way could be considerably greater or less than the actual yield.

- 3. Critically evaluate the yield functions presented below in terms of isotropy, pressure-dependence, and existence of a Bauschinger effect. Demonstrate your results. Why would you choose to use each yield function?
 - a. Hill's¹ Quadratic Yield Function (1948) $f = F(\sigma_2 - \sigma_3)^2 + G(\sigma_1 - \sigma_3)^2 + H(\sigma_1 - \sigma_2)^2$
 - b. Hill's² Non-Quadratic Yield Function (1979) $f = F \left| \sigma_2 \sigma_3 \right|^{M} + G \left| \sigma_1 \sigma_3 \right|^{M} + H \left| \sigma_1 \sigma_2 \right|^{M}$
 - c. Bourne and Hill³ ($\sigma_3 = 0$, not principal axes): $f = 3\sigma_x^3 - 6\sigma_x^2\sigma_y - 6\sigma_x\sigma_y^2 + 4\sigma_y^3 + (4\sigma_x + 21\sigma_y)\sigma_{xy}^2$

R. Hill: <u>The Mathematical Theory of Plasticity</u>, Clarendon Press, 1950, p. 318.
 R. Hill: <u>Proc. Roy. Soc.</u>, 1949, vol. 198, p. 428.

R. Hill: <u>Math. Proc. Camb. Philos. Soc.</u>, 1979, vol. 85, p. 179.
 P.B. Mellor and A. Parmar, <u>Mechanics of Sheet Metal Forming</u>, D.P. Koistinen and N.M. Wang, Plenum Press, 1978, p.67.

³ L. Bourne and R. Hill: <u>Philos. Mag.</u>, 1950, vol. 41, p. 671.

- *d.* Drucker⁴: $f = (J_2)^3 c (J_3)^2$, where c = constant
- e. Edelman and Drucker⁵: $f = \frac{1}{2} c_{ijkl} \left(\sigma_{ij}^{'} - M \epsilon_{ij}^{P} \right) \left(\sigma_{kl}^{'} - M \epsilon_{kl}^{P} \right), \quad c_{ijkl}, M = \text{constants}, \ \epsilon_{kl} = \text{plastic strain}$
- f. Gotoh⁶ ($\sigma_3 = 0$): $f = A_o \left(\sigma_x + \sigma_y\right)^4 + A_1 \sigma_x^4 + A_2 \sigma_x^3 \sigma_y + A_3 \sigma_x^2 \sigma_y^2 + A_4 \sigma_x \sigma_y^3 + A_5 \sigma_y^4 + \left(A_6 \sigma_x^2 + A_7 \sigma_x \sigma_y + A_8 \sigma_y^2\right) \sigma_{xy}^2 + A_9 \sigma_{xy}^4, \quad A_1 = \text{constants.}$
- g. Bassani⁷ ($\sigma_3 = 0$): f = $|\sigma_1 + \sigma_2|^N + (1 + 2r) k |\sigma_1 - \sigma_2|^M$, r, k, M = constants.
- h. Jones & Gillis⁸ ($\sigma_3 = 0$): f = $c_{11}\sigma_x^2 + c_{12}\sigma_x\sigma_y + c_{13}\sigma_x\sigma_{xy} + c_{22}\sigma_y^2 + c_{23}\sigma_y\sigma_{xy} + c_{33}\tau_{xy}^2$, c_{ij} = constants

i. Gupta⁹:
$$f = \sqrt{J_2} - \alpha_p J_1$$
, $\alpha = constant$.

SOLUTION:

a. • Not isotropic (unless F = G = H)

• Pressure-independent [since
$$f(\sigma_i + p) = f(\sigma_j)$$
]

- No Bauschinger effect [since $f(\sigma_i) = f(-\sigma_i)$]
- Modification of von Mises to handle simple anisotropy for example, for rolled sheet. Allows different strains in two lateral directions in tensile test.
- b. Not isotropic (unless F = G = H)
 - Pressure-independent
 - No Bauschinger effect
 - Includes the anisotropy of Part a., but allows further adjustment of shape to account for higher or lower flow strength in plane-strain or balanced biaxial tension (relative to uniaxial tension) for a given r value.
- c. Not isotropic
 - Cannot determine pressure-dependence since only two principal stresses are represented
 - A Bauschinger effect is inherent (because of the cubic terms)
 - This is a purely 2-D yield function with fixed constants that was fit to experimental data from

- 5 F. Edelman and D.C. Drucker: J. Franklin Inst., 1951, vol. 251, p. 581.
- 6 M. Gotoh: Int. J. Mech. Sci., 1977, vol. 19, p.505.

8 (unpublished)

⁴ D.C. Drucker: <u>J. Appl. Mech.</u>, <u>Trans. ASME</u>, 1949, vol. 16, p. 349.

⁷ J.L. Bassani: Int. J. Mech. Sci., 1977, vol. 19, p. 651.

⁹ Y.M. Gupta: <u>Acta Metall.</u>, 1977, vol. 25, p. 1509.

tensile tests conducted at various angles in a sheet of metal. It provides a detailed variation of σ_y vs. θ but has no consistent 3-D form.

- Isotropic (use of invariants proves this) d. •
 - Pressure-independent (use of deviatoric components demonstrates this)
 - No Bauschinger effect (the odd invariant, J_3^{-1} , has been squared to avoid this.
 - This is a very interesting generalization of the simplest yield function, $J_2' = k$, which satisfies pressure-independence, isotropy, and no Bauschinger effect. Because it is isotropic, it cannot contribute to fitting strain ratio effects on sheet tensile tests, however. It is similar in concept to Part b., with F = G = H.
- Not isotropic (unless $C_{ijkl} = C_{mnop}$, M = 0, or $\varepsilon_{ij} = 0$) e. •
 - Pressure-independent (from use of deviatoric components)
 - Bauschinger effect (unless M = 0)
 - This is the logical extension to the most general quadratic yield function in terms of anisotropy, with kinematic hardening depending on total plastic strain.
- f. Not isotropic
 - Cannot determine pressure-dependence since only two principal stresses are represented ٠
 - A Bauschinger effect is inherent (because of the cubic power terms)
 - This is a purely 2-D yield function with fixed constants that was fit to experimental data from tensile tests conducted at various angles in a sheet of metal. It provides a detailed variation of σ_y vs. θ but has not consistent 3-D form.

- Isotropic (cannot distinguish σ_1 and σ_2 because of absolute values) g. •
 - Cannot determine pressure dependence since only two principal stresses are represented ٠
 - A Bauschinger effect is present (unless N and M are even)
 - Very similar to Part b., but the different power for the two terms means it must be handled numerically.
- See Parts c. and f. h. •
- i. Isotropic (use of invariants)
 - Pressure-dependent (unless $\alpha_p = 0$) •
 - Bauschinger effect (unless $\alpha_p = 0$)
 - This yield function is the simplest one which includes a pressure dependence, which is its purpose. For example, compressible materials such as powders or sponges could be treated with this function.
- Write each of the yield functions in Problem 3 in terms of each of the following definitions of 4. effective stress:
 - a. tensile test in the x_1 direction, $\overline{\sigma} = \sigma_1$
 - *b. tensile test in the* x_2 *direction*, $\overline{\sigma} = \sigma_2$

- *c.* balanced biaxial test, $\sigma_1 = \sigma_2 = \overline{\sigma}$
- *d.* shear test, same principal axes, $\sigma_1 = \bar{\sigma}$, $\sigma_2 = -\bar{\sigma}$

SOLUTION:

b.

d.

a. x_1 tension: $(G + H) \bar{\sigma}^2 = f$

- x_2 tension: $(F + H) \bar{\sigma}^2 = f$
- Bal. Biax: $(F+G) \overline{\sigma}^2 = f$

Shear (assume that $\tau = \tau_{xy}$ occurs 45° from principal axes, i.e. as stress state consisting of τ_{xy} is equivalent to $\sigma_1 = \overline{\sigma}$, $\sigma_2 = -\overline{\sigma}$.

$$\left(F+G+4H\right)\overline{\sigma}^{\,2}=f$$

$$\begin{aligned} x_{1} \text{ tension:} &= (G + H) \,\overline{\sigma}^{M} = f \\ x_{2} \text{ tension:} & (F + H) \,\overline{\sigma}^{M} = f \\ \text{Bal. Biax:} & (F + G) \,\overline{\sigma}^{M} = f \\ \text{Shear:} & (F + G + 2^{M}H) \,\overline{\sigma}^{M} \quad (\text{where } \sigma_{1} = \overline{\sigma}, \ \sigma_{2} = -\overline{\sigma}) \end{aligned}$$

c. x_1 tension: $3 \bar{\sigma}^3 = f$

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x_2 tension: 4 \bar{\sigma}^3 = f
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Bal. Biax: -5 \overline{\sigma}^3 = f
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Shear: $-\bar{\sigma}^3 = f$ (where $\sigma_1 = \bar{\sigma}, \sigma_2 = -\bar{\sigma}$)

$$\mathbf{x}_{1} \text{ tension:} \quad \begin{bmatrix} \boldsymbol{\sigma} \end{bmatrix} = \begin{bmatrix} \boldsymbol{\bar{\sigma}} & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}, \quad \begin{bmatrix} \boldsymbol{\sigma}' \end{bmatrix} = \begin{bmatrix} \frac{2}{3}\boldsymbol{\bar{\sigma}} & 0 & 0 \\ 0 & -\frac{1}{3}\boldsymbol{\bar{\sigma}} & 0 \\ 0 & 0 & -\frac{1}{3}\boldsymbol{\bar{\sigma}} \end{bmatrix}$$

e.

$$J_{2}' = -\left(\sigma_{11}' \sigma_{22}' + \sigma_{22}' \sigma_{33}' + \sigma_{11}' \sigma_{33}'\right) = -\left[\frac{2}{3}\left(-\frac{1}{3}\right)\bar{\sigma}^{2} + \left(-\frac{1}{3}\right)\left(-\frac{1}{3}\right)\bar{\sigma}^{2} + \left(\frac{2}{3}\right)\left(-\frac{1}{3}\right)\bar{\sigma}^{2}\right] = \frac{1}{3}\bar{\sigma}^{2}$$
$$J_{3}' = \sigma_{11}' \sigma_{22}' \sigma_{33}' = \frac{2}{27}\bar{\sigma}^{3}$$
$$\frac{1}{27}\bar{\sigma}^{6} - C\frac{4}{729}\bar{\sigma}^{6} = \frac{1}{27}\left(1 - \frac{4}{27}C\right)\bar{\sigma}^{6} = f$$

 x_2 tension: Same as x_1 because of isotropy

Bal. Biax:
$$\begin{bmatrix} \sigma \end{bmatrix} = \begin{bmatrix} \overline{\sigma} & 0 & 0 \\ 0 & \overline{\sigma} & 0 \\ 0 & 0 & 0 \end{bmatrix}, \begin{bmatrix} \sigma' \end{bmatrix} = \begin{bmatrix} \frac{1}{3}\overline{\sigma} & 0 & 0 \\ 0 & \frac{1}{3}\overline{\sigma} & 0 \\ 0 & 0 & -\frac{2}{3}\overline{\sigma} \end{bmatrix}$$

This is the same as x_1 and x_2 tension because of a) isotropy and b) no Baushinger effect, so that a stress state of σ_{ij} is equivalent to $-\sigma_{ij}$

Shear:
$$\begin{bmatrix} \sigma \end{bmatrix} = \begin{bmatrix} \overline{\sigma} & 0 & 0 \\ 0 & -\overline{\sigma} & 0 \\ 0 & 0 & 0 \end{bmatrix}$$
, $\begin{bmatrix} \sigma \end{bmatrix} = \begin{bmatrix} \overline{\sigma} & 0 & 0 \\ 0 & -\overline{\sigma} & 0 \\ 0 & 0 & 0 \end{bmatrix}$

$$J_{2}' = -(-\overline{\sigma}^{2} + 0 + 0) = \overline{\sigma}^{2}$$

$$J_{3}' = \sigma_{11}' \sigma_{22}' \sigma_{33}' = 0, \text{ so } \overline{\sigma}^{6} = f \quad \text{[same as von Mises]}$$

$$x_{1} \text{ tension: } \begin{bmatrix} \sigma' \end{bmatrix} = \begin{bmatrix} \frac{2}{3}\overline{\sigma} & 0 & 0 \\ 0 & -\frac{1}{3}\overline{\sigma} & 0 \\ 0 & 0 & -\frac{1}{3}\overline{\sigma} \end{bmatrix}$$

For simplicity, we assume no initial loading, such that ϵ_{ij} = 0, then:

$$\frac{1}{2} \left[C_{1111} \left(\frac{2}{3} \right) \left(\frac{2}{3} \right) \bar{\sigma}^2 + C_{1122} \left(\frac{2}{3} \right) \left(-\frac{1}{3} \right) \bar{\sigma}^2 + C_{1133} \left(\frac{2}{3} \right) \left(-\frac{1}{3} \right) \bar{\sigma}^2 + C_{2211} \left(-\frac{1}{3} \right) \left(\frac{2}{3} \right) \bar{\sigma}^2 + C_{2222} \left(-\frac{1}{3} \right) \left(-\frac{1}{3} \right) \bar{\sigma}^2 + C_{2233} \left(-\frac{1}{3} \right) \left(-\frac{1}{3} \right) \bar{\sigma}^2 + C_{3311} \left(-\frac{1}{3} \right) \left(\frac{2}{3} \right) \bar{\sigma}^2 + C_{3322} \left(-\frac{1}{3} \right) \left(-\frac{1}{3} \right) \bar{\sigma}^2 + C_{3333} \left(-\frac{1}{3} \right) \left(-\frac{1}{3} \right) \bar{\sigma}^2 \right] = f$$

$$\frac{\overline{\sigma}^2}{18} \left[4 C_{1111} - 2 \left(C_{1122} + C_{1133} + C_{2211} + C_{3311} \right) + \left(C_{2222} + C_{2233} + C_{3322} + C_{3333} \right) \right] = f$$

x₂tension:

$$\frac{\overline{\sigma}^2}{18} \left[4 C_{2222} - 2 \left(C_{1122} + C_{2211} + C_{3311} + C_{1133} \right) + \left(C_{1111} + C_{3333} + C_{1133} + C_{3311} \right) \right] = f$$

Bal. Biax:
$$\frac{\overline{\sigma}^2}{18} \left[4 C_{3333} - 2 \left(C_{1133} + C_{3311} + C_{2233} + C_{3322} \right) + \left(C_{1111} + C_{2222} + C_{1122} + C_{2211} \right) \right] = f$$

 $= -\overline{\sigma}$

Shear:
$$\frac{\bar{\sigma}^2}{2} \left[C_{1111} + C_{2222} - C_{1122} - C_{2211} \right] = f$$
, where: $\sigma_1 = \bar{\sigma}, \sigma_2 = -\bar{\sigma}$

$$\mathbf{x}_1$$
 tension: $\overline{\mathbf{\sigma}}^4 \left(\mathbf{A}_0 + \mathbf{A}_1 \right) = \mathbf{f}$

$$x_{2} \text{ tension:} \quad \overline{\sigma}^{4} \left(A_{0} + A_{5} \right) = f$$

Bal. Biax:
$$\overline{\sigma}^{4} \left[16 A_{0} + A_{1} + A_{2} + A_{3} + A_{4} + A_{5} \right] = f$$

Shear:
$$\overline{\sigma}^{4} \left[A_{1} - A_{2} + A_{3} - A_{4} + A_{5} \right] = f, \text{ where: } \sigma_{1} = \overline{\sigma}, \ \sigma_{2}$$

$$\mathbf{x}_1$$
 tension: $\mathbf{\bar{\sigma}}^{N} + (1+2\mathbf{r}) \mathbf{K} \mathbf{\bar{\sigma}}^{M} = \mathbf{f}$

$$\mathbf{x}_2$$
 tension: $\mathbf{\bar{\sigma}}^{N} + (1+2\mathbf{r}) \mathbf{K} \mathbf{\bar{\sigma}}^{M} = \mathbf{f}$

Bal. Biax: $2^{N} \overline{\sigma}^{N} = f$

Shear:
$$(1+2\mathbf{r}) \mathbf{k} 2^{\mathbf{M}} \overline{\mathbf{\sigma}}^{\mathbf{M}} = \mathbf{f}$$
, where: $\sigma_1 = \overline{\mathbf{\sigma}}, \sigma_2 = -\overline{\mathbf{\sigma}}$

h.
$$x_1$$
 tension: $c_{11}\overline{\sigma}^2 = f$

 \mathbf{x}_2 tension: $\mathbf{c}_{22}\mathbf{\bar{\sigma}}^2 = \mathbf{f}$

Bal. Biax: $c_{11} \bar{\sigma}^2 + c_{12} \bar{\sigma}^2 + c_{22} \bar{\sigma}^2 = f$

Shear: $c_{11}\overline{\sigma}^2 - c_{12}\overline{\sigma}^2 + c_{22}\overline{\sigma}^2 = f$, where: $\sigma_1 = \overline{\sigma}$, $\sigma_2 = -\overline{\sigma}$

i. \mathbf{X}_1 tension: $\mathbf{J}_2' = \frac{1}{3} \,\overline{\sigma}^2 \quad \mathbf{J}_1 = \overline{\sigma}$, therefore: $\frac{1}{\sqrt{3}} \,\overline{\sigma} - \alpha_p \,\overline{\sigma} = \left(\frac{1}{\sqrt{3}} - \alpha_p\right) \overline{\sigma} = \mathbf{f}$

 x_2 tension: (same as x_1 , isotropic)

Bal. Biax:
$$J_2' = \frac{1}{3} \overline{\sigma}^2$$
, $J_1 = 2\overline{\sigma}$, therefore: $(\frac{1}{\sqrt{3}} - 2\alpha_p)\overline{\sigma} = f$
Shear: $J_2' = \overline{\sigma}^2$ $J_1 = 0$, therefore: $\overline{\sigma} = f$, where: $\sigma_1 = \overline{\sigma}$, $\sigma_2 = -$

5. Derive the normality condition for each of the yield functions in Problem 3.

SOLUTION:

a.

c.

d.

$$d\varepsilon_{2} = 2d\lambda \left[\left(F + H \right) \sigma_{2} - H\sigma_{1} - F\sigma_{3} \right]$$
$$d\varepsilon_{3} = 2d\lambda \left[\left(F + G \right) \sigma_{3} - G\sigma_{1} - F\sigma_{2} \right]$$

 $d\varepsilon_{1} = 2d\lambda \left[\left(G + H \right) \sigma_{1} - H \sigma_{2} - G \sigma_{3} \right]$

b. Assume that $\sigma_1 > \sigma_2 > \sigma_3 > 0$ (other cases can be handled separately), then:

$$\begin{split} d\epsilon_{1} &= Md\lambda \left[G\left(\sigma_{1} - \sigma_{3}\right)^{M-1} + H\left(\sigma_{1} - \sigma_{2}\right)^{M-1} \right] \\ d\epsilon_{2} &= Md\lambda \left[F\left(\sigma_{2} - \sigma_{3}\right)^{M-1} - H\left(\sigma_{1} - \sigma_{2}\right)^{M-1} \right] \\ d\epsilon_{3} &= Md\lambda \left[-F\left(\sigma_{2} - \sigma_{3}\right)^{M-1} - G\left(\sigma_{1} - \sigma_{3}\right)^{M-1} \right] \\ d\epsilon_{x} &= d\lambda \left[9 \sigma_{x}^{2} - 12 \sigma_{x} \sigma_{y} - 6 \sigma_{y}^{2} + 4 \sigma_{xy}^{2} \right] \\ d\epsilon_{y} &= d\lambda \left[-6 \sigma_{x}^{2} - 12 \sigma_{x} \sigma_{y} + 12 \sigma_{y}^{2} + 21 \sigma_{xy}^{2} \right] \\ d\epsilon_{xy} &= d\lambda \left[(4 \sigma_{x} + 21 \sigma_{y}) 2\sigma_{xy} \right] \\ d\epsilon_{1} &= d\lambda \frac{\partial f}{\partial \sigma_{1}} = d\lambda \left[3 \left(J_{2} \right)^{2} \frac{\partial J_{2}}{\partial \sigma_{1}} - 2 c J_{3} \cdot \frac{\partial J_{3}}{\partial \sigma_{1}} \right] = d\lambda \left[3 \left(J_{2} \right)^{2} \sigma_{1} - \frac{2}{3} c J_{3} \cdot \left(J_{2} + 3\sigma_{2} \cdot \sigma_{3} \right) \right] \end{split}$$

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$$= d\lambda \left[(J_{2})^{2} (2\sigma_{1} - \sigma_{2} - \sigma_{3}) - \frac{2}{9} c J_{3}' (2\sigma_{1}^{2} - \sigma_{2}^{2} - \sigma_{3}^{2} - 2\sigma_{1}\sigma_{2} - 2\sigma_{1}\sigma_{3} + 4\sigma_{2}\sigma_{3}) \right]$$

$$d\epsilon_{2} = d\lambda \frac{\partial f}{\partial \sigma_{2}} = d\lambda \left[3 (J_{2})^{2} \frac{\partial J_{2}'}{\partial \sigma_{2}} - 2 c J_{3}' \frac{\partial J_{3}'}{\partial \sigma_{2}} \right] = d\lambda \left[3 (J_{2})^{2} \sigma_{2}' - \frac{2}{3} c J_{3}' (J_{2}' + 3\sigma_{1}'\sigma_{3}') \right]$$

$$= d\lambda \left[(J_{2})^{2} (2\sigma_{2} - \sigma_{1} - \sigma_{3}) - \frac{2}{9} c J_{3}' (2\sigma_{2}^{2} - \sigma_{1}^{2} - \sigma_{3}^{2} - 2\sigma_{1}\sigma_{2} + 4\sigma_{1}\sigma_{3} - 2\sigma_{2}\sigma_{3}) \right]$$

$$d\epsilon_{3} = d\lambda \frac{\partial f}{\partial \sigma_{3}} = d\lambda \left[3 (J_{2})^{2} \frac{\partial J_{2}'}{\partial \sigma_{3}} - 2 c J_{3}' \frac{\partial J_{3}'}{\partial \sigma_{3}} \right] = d\lambda \left[3 (J_{2})^{2} \sigma_{3}' - \frac{2}{3} c J_{3}' (J_{2}' + 3\sigma_{1}'\sigma_{2}') \right]$$

$$= d\lambda \left[(J_{2})^{2} (2\sigma_{3} - \sigma_{1} - \sigma_{2}) - \frac{2}{9} c J_{3}' (2\sigma_{3}^{2} - \sigma_{1}^{2} - \sigma_{2}^{2} + 4\sigma_{1}\sigma_{2} - 2\sigma_{1}\sigma_{3} - 2\sigma_{2}\sigma_{3}) \right]$$

where, from Eqs. 3.34 and 3.35:

$$J_{2}' = \frac{1}{6} \left[\left(\sigma_{1} - \sigma_{2} \right)^{2} + \left(\sigma_{1} - \sigma_{3} \right)^{2} + \left(\sigma_{2} - \sigma_{3} \right)^{2} \right] = \frac{1}{3} \left(\sigma_{1}^{2} + \sigma_{2}^{2} + \sigma_{3}^{2} \right) - \frac{1}{3} \left(\sigma_{1} \sigma_{2} + \sigma_{1} \sigma_{3} + \sigma_{2} \sigma_{3} \right)$$

$$J_{3}' = \sigma_{1}' \sigma_{2}' \sigma_{3}' = \frac{1}{27} \left(2\sigma_{1} - \sigma_{2} - \sigma_{3} \right) \left(2\sigma_{2} - \sigma_{1} - \sigma_{3} \right) \left(2\sigma_{3} - \sigma_{1} - \sigma_{2} \right)$$

$$= \frac{2}{27} \left(\sigma_{1}^{3} + \sigma_{2}^{3} + \sigma_{3}^{3} \right) - \frac{1}{9} \left(\sigma_{1} \sigma_{2}^{2} + \sigma_{1}^{2} \sigma_{2} + \sigma_{1} \sigma_{3}^{2} + \sigma_{1}^{2} \sigma_{3} + \sigma_{2} \sigma_{3}^{2} + \sigma_{2}^{2} \sigma_{3} \right) + \frac{4}{9} \sigma_{1} \sigma_{2} \sigma_{3}$$

e. Consider only the start of deformation, when $\epsilon_{ij}^{P} = 0$, and write in principal axes.

$$f = C_{ij} \sigma_{i} \sigma_{j}, \quad \frac{\partial f}{\partial \sigma_{m}} = C_{ij} \frac{\partial}{\partial \sigma_{m}} \sigma_{i} \sigma_{j} = C_{ij} \left(\sigma_{i} \frac{\partial \sigma_{j}}{\partial \sigma_{m}} + \sigma_{j} \frac{\partial \sigma_{i}}{\partial \sigma_{m}} \right)$$

but note that since $\sigma_{i} = 2\sigma_{i} - \sigma_{j} - \sigma_{k}$, where $i \neq j \neq k$, that $\frac{\partial \sigma_{j}}{\partial \sigma_{m}} = \begin{cases} 2 & \text{if } j = m \\ -1 & \text{if } j \neq m \end{cases}$

Therefore, we can write:

$$\begin{aligned} \frac{d\varepsilon_{1}}{d\lambda} &= 2\left(2C_{11} + C_{12} + C_{13} + C_{21} + C_{31}\right)\sigma_{1}' \\ &- 1\left(2C_{22} + C_{21} + C_{23} + C_{12} + C_{13}\right)\sigma_{2}' \\ &- 1\left(2C_{33} + C_{31} + C_{32} + C_{13} + C_{23}\right)\sigma_{3}' \\ &, \text{ or } \end{aligned}$$

h.

i.

$$\frac{d\varepsilon_{1}}{d\lambda} = \frac{2}{3} \left(2C_{11} + C_{12} + C_{13} + C_{21} + C_{31} \right) \left(2\sigma_{1} - \sigma_{2} - \sigma_{3} \right) - \frac{1}{3} \left(2C_{22} + C_{21} + C_{23} + C_{12} + C_{13} \right) \left(2\sigma_{2} - \sigma_{1} - \sigma_{3} \right) - \frac{1}{3} \left(2C_{33} + C_{31} + C_{32} + C_{13} + C_{23} \right) \left(2\sigma_{3} - \sigma_{1} - \sigma_{2} \right)$$

The other components follow by inspection. (For example $d\epsilon_2$ would look the same except that the coefficients on the three terms would be, respectively, $-\frac{1}{3}, \frac{2}{3}, -\frac{1}{3}$, or for $d\epsilon_2$ the coefficients would become $-\frac{1}{3}, -\frac{1}{3}, \frac{2}{3}$.

f.

$$\frac{d\varepsilon_{x}}{d\lambda} = 4A_{o}\left(\sigma_{x} + \sigma_{y}\right)^{3} + 4A_{1}\sigma_{x}^{3} + 3A_{2}\sigma_{x}^{2}\sigma_{y} + 2A_{3}\sigma_{x}\sigma_{y}^{2} + A_{4}\sigma_{y}^{3} + 2A_{6}\sigma_{x}\sigma_{xy}^{2} + A_{7}\sigma_{y}\sigma_{xy}^{2} \\
\frac{d\varepsilon_{y}}{d\lambda} = 4A_{o}\left(\sigma_{x} + \sigma_{y}\right)^{3} + A_{2}\sigma_{x}^{3} + 2A_{3}\sigma_{x}^{2}\sigma_{y} + 3A_{4}\sigma_{x}\sigma_{y}^{2} + 4A_{5}\sigma_{y}^{3} + A_{7}\sigma_{x}\sigma_{xy}^{2} + 2A_{8}\sigma_{y}\sigma_{xy}^{2} \\
\frac{d\varepsilon_{xy}}{d\lambda} = 2\left(A_{6}\sigma_{x}^{2} + A_{7}\sigma_{x}\sigma_{y} + A_{8}\sigma_{y}^{2}\right)\sigma_{xy} + 4A_{9}\sigma_{xy}^{3}$$

g. Assume that $\sigma_1 > \sigma_2 > 0$ for simplicity. Other cases may be derived separately.

$$d\epsilon_{1} = d\lambda \bigg[N(\sigma_{1} + \sigma_{2})^{N-1} + (1 + 2t) k M(\sigma_{1} - \sigma_{2})^{M-1} \bigg]$$

$$d\epsilon_{2} = d\lambda \bigg[N(\sigma_{1} + \sigma_{2})^{N-1} - (1 + 2t) M(\sigma_{1} - \sigma_{2})^{M-1} \bigg]$$

$$d\epsilon_{x} = d\lambda \bigg[2c_{11} \sigma_{x} + c_{12} \sigma_{y} + c_{13} \sigma_{xy} \bigg]$$

$$d\epsilon_{y} = d\lambda \bigg[c_{12} \sigma_{x} + 2c_{22} \sigma_{y} + c_{23} \sigma_{xy} \bigg]$$

$$d\epsilon_{xy} = d\lambda \bigg[c_{13} \sigma_{x} + 2c_{23} \sigma_{y} + 2c_{33} \sigma_{xy} \bigg]$$

$$J_{2}' = + \frac{1}{6} \bigg[(\sigma_{1}^{2} - \sigma_{2}^{2})^{2} + (\sigma_{1}^{2} - \sigma_{3}^{2})^{2} + (\sigma_{2}^{2} - \sigma_{3}^{2})^{2} \bigg], J_{1} = \sigma_{1} + \sigma_{2} + \sigma_{3}$$

$$\mathbf{f} = (\mathbf{J}_2)^{\frac{1}{2}} - \boldsymbol{\alpha}_p \mathbf{J}_1 \quad \text{, therefore:} \quad \frac{\partial \mathbf{f}}{\partial \sigma_1} = \frac{1}{2} (\mathbf{J}_2)^{-\frac{1}{2}} \left(\frac{\partial \mathbf{J}_2}{\partial \sigma_1} \right) - \boldsymbol{\alpha}_p \frac{\partial \mathbf{J}_1}{\partial \sigma_1}$$

$$\frac{\mathrm{d}\varepsilon_{1}}{\mathrm{d}\lambda} = \frac{\left(2\,\sigma_{1}-\sigma_{2}-\sigma_{3}\right)}{6\,\sqrt{J_{2}^{'}}} - \alpha_{p} = \frac{\left(2\,\sigma_{1}-\sigma_{2}-\sigma_{3}\right)}{\left[\left(\sigma_{1}-\sigma_{2}\right)^{2}+\left(\sigma_{2}-\sigma_{3}\right)^{2}+\left(\sigma_{1}-\sigma_{3}\right)^{2}\right]^{\frac{1}{2}}} - \alpha_{p}$$

 $(d\epsilon_2, d\epsilon_3 \text{ follow directly})$

6. Assume that $\sigma_{i3} = 0$ and that we are working in principal axes for each yield function in Problem 3. Letting $\alpha = \sigma_2/\sigma_1$ and $\beta = d\epsilon_2/d\epsilon_1$, find expressions for $\alpha(\beta)$ and for $\beta(\alpha)$.

SOLUTION:

$$\beta = \frac{d\varepsilon_2}{d\varepsilon_1} = \frac{(F+H)\sigma_2 - H\sigma_1}{(G+H)\sigma_1 - H\sigma_2} = \frac{(F+H)\alpha - H}{(G+H) - H\alpha}$$

a.

$$\alpha = \frac{\sigma_2}{\sigma_1} = \frac{\beta \left(G + H\right) + H}{F + H + \beta H}$$

b. Assuming that
$$\sigma_1 > \sigma_2 > 0$$
:
$$\beta = \frac{d\varepsilon_2}{d\varepsilon_1} = \frac{F \sigma_2^{M-1} - H (\sigma_1 - \sigma_2)^{M-1}}{G \sigma_1^{M-1} + H (\sigma_1 - \sigma_2)^{M-1}} = \frac{F \alpha^{M-1} - H (1 - \alpha)^{M-1}}{G \alpha^{M-1} + H (1 - \alpha)^{M-1}}$$

 $\alpha = ?$ (Because of the form, α (B) is transcendental, so α (B) must be found numerically.)

,

c.

$$\beta = \frac{d\varepsilon_y}{d\varepsilon_x} = \frac{-6 - 12 \alpha + 12 \alpha^2}{9 - 12 \alpha - 6 \alpha^2}$$

$$\alpha = \frac{-2(\beta-1) \pm \sqrt{10\beta^2 + 8\beta + 12}}{2(\beta+2)}$$

which is found by using the quadratic formula on $\beta(\alpha)$.

$$d\epsilon_{1} = d\lambda \left[\frac{1}{3} \left(\sigma_{1}^{2} + \sigma_{2}^{2} - \sigma_{1} \sigma_{2} \right) \left(2 \sigma_{1} - \sigma_{2} \right) - \frac{2}{243} \left(2 \sigma_{1} - \sigma_{2} \right) \right], \text{ or}$$

$$= d\lambda \left[\left(2 \sigma_{2} - \sigma_{1} \right) \left(\sigma_{1} + \sigma_{2} \right) \left(2 \sigma_{1}^{2} - \sigma_{2}^{2} - 2 \sigma_{1} \sigma_{2} \right) \right]$$

$$d\epsilon_{2} = d\lambda \left[\frac{1}{3} \left(\sigma_{1}^{2} + \sigma_{2}^{2} - \sigma_{1} \sigma_{2} \right) \left(2 \sigma_{2} - \sigma_{1} \right) - \frac{2}{243} \left(2 \sigma_{1} - \sigma_{2} \right) \right], \text{ or}$$

$$= d\lambda \left[\left(2 \sigma_2 - \sigma_1 \right) \left(\sigma_1 + \sigma_2 \right) \left(2 \sigma_2^2 - \sigma_1^2 - 2 \sigma_1 \sigma_2 \right) \right]$$

$$\beta = \frac{d\varepsilon_2}{d\varepsilon_1} = \frac{\left(2\alpha - 1 \right) \left[\left(1 + \alpha^2 - \alpha \right) - \frac{2}{81} \left(2 - \alpha \right) \left(1 + \alpha \right) \left(2 \alpha^2 - 1 - 2\alpha \right) \right]}{\left(2 - \alpha \right) \left[\left(1 + \alpha^2 - \alpha \right) - \frac{2}{81} \left(2\alpha - 1 \right) \left(1 + \alpha \right) \left(2 - \alpha^2 - 2\alpha \right) \right]}$$

 α = hopeless to do in closed form unless the numerator and denomerator can be factored.

$$\beta = \frac{d\epsilon_2}{d\epsilon_1} = \frac{\left(c_{12} + c_{21}\right)\left(2\sigma_1 - \sigma_2\right) + c_{22}\left(2\sigma_2 - \sigma_1\right) + \left(c_{23} + c_{32}\right)\left(-\sigma_1 - \sigma_2\right)}{2 c_{11}\left(2\sigma_1 - \sigma_2\right) + \left(c_{12} + c_{21}\right)\left(2\sigma_2 - \sigma_1\right) + \left(c_{13} + c_{31}\right)\left(-\sigma_1 - \sigma_2\right)}$$

$$\beta = \frac{\left(c_{12} + c_{21}\right)\left(2 - \alpha\right) + c_{22}\left(2\alpha - 1\right) + \left(c_{23} + c_{32}\right)\left(-1 - \alpha\right)}{2 c_{11}\left(2 - \alpha\right) + \left(c_{12} + c_{21}\right)\left(2\alpha - 1\right) + \left(c_{13} + c_{31}\right)\left(-1 - \alpha\right)}$$

$$\alpha = \frac{2\left(c_{12} + c_{21} - c_{22}\right) - \left(c_{23} - c_{32}\right) + \beta\left(-4 c_{11} + c_{12} + c_{21} + c_{13} + c_{31}\right)}{c_{12} + c_{21} + c_{23} + c_{32} - 2 c_{22} + \beta\left[-2 c_{11} + 2\left(c_{12} + c_{21}\right) - c_{13} - c_{31}\right]}$$

$$\beta = \frac{d\epsilon_2}{d\epsilon_1} = \frac{4 A_0 \left(1 + \alpha\right)^3 + A_2 + 2 A_3 \alpha + 3 A_4 \alpha^2 + 4 A_5 \alpha^3}{4 A_0 \left(1 + \alpha\right)^3 + 4 A_1 + 3 A_2 \alpha + 2 A_3 \alpha^2 + A_4 \alpha^3}$$

f.

e.

 $\alpha(\beta)\,$ is a cubic equation and may be solved numerically or in closed form.

$$\beta = \frac{d\varepsilon_2}{d\varepsilon_1} = \frac{N\left(1+\alpha\right)^{N-1} - \left(1+2r\right)M\left(1-\alpha\right)^{M-1}}{N\left(1+\alpha\right)^{N-1} + \left(1+2r\right)M\left(1-\alpha\right)^{M-1}}$$

g.

 $\alpha(\beta)$ = transcendental

 $\beta = \frac{d\epsilon_2}{d\epsilon_1} = \frac{c_{12} + 2c_{22}\alpha}{2c_{11} + c_{12}\alpha}$

h.

$$\alpha = \frac{c_{12} - 2\beta c_{11}}{\beta c_{12} - 2c_{22}}$$

i.

$$d\varepsilon_{1} = \frac{2\sigma_{1} - \sigma_{2}}{\sqrt{2\left(\sigma_{1}^{2} + \sigma_{2}^{2} - \sigma_{1}\sigma_{2}\right)}} - \alpha_{p}, \text{ and } d\varepsilon_{2} = \frac{2\sigma_{2} - \sigma_{1}}{\sqrt{2\left(\sigma_{1}^{2} + \sigma_{2}^{2} - \sigma_{1}\sigma_{2}\right)}} - \alpha_{p}$$
$$\beta = \frac{d\varepsilon_{2}}{d\varepsilon_{1}} = \frac{2\alpha - 1 - \alpha_{p}\sqrt{2\left(\sigma_{1}^{2} + \sigma_{2}^{2} - \sigma_{1}\sigma_{2}\right)}}{2 - \alpha - \alpha_{p}\sqrt{2\left(\sigma_{1}^{2} + \sigma_{2}^{2} - \sigma_{1}\sigma_{2}\right)}}$$

 $\alpha(\beta)$ appears to be transcendental, and thus must be found numerically.

- 7. Construct a full set of useful equations expressed in principal axes for the following yield functions. Useful equations include the yield function in terms of $\overline{\sigma}$, the associated flow rule, the normality equations (forward and inverse), definition of $d\lambda$ in terms of $\overline{\sigma}$ and $d\overline{\varepsilon}$ (where x_1 tension is the standard state), $d\overline{\varepsilon}$ in terms of $d\varepsilon_1$, $d\varepsilon_2$ and $d\varepsilon_3$, $\alpha(\beta)$, and $\beta(\alpha)$; where $\alpha = \sigma_2/\sigma_1$ and $\beta = d\varepsilon_2/d\varepsilon_1$ (The last sets of equations were derived in Problem 6.)
 - a. von Mises: $f = (\sigma_1 \sigma_2)^2 + (\sigma_1 \sigma_3)^2 + (\sigma_2 \sigma_3)^2$
 - b. Hill quadratic: $f = F(\sigma_2 \sigma_3)^2 + G(\sigma_1 \sigma_3)^2 + H(\sigma_1 \sigma_2)^2$
 - *c. Hill normal anisotropic: modify function #2 such that the* x_1 *and* x_2 *axes are equivalent and the strain ratio* $d\epsilon_2/d\epsilon_1$ *in an* x_1 *tensile test is r.* (x_3 *is the sheet-thickness direction*).
 - *d. Hill non-quadratic (Case IV),* $(\sigma_3 = 0)$:: $\mathbf{f} = (1+2\mathbf{r}) |\sigma_1 \sigma_2|^M + |\sigma_1 + \sigma_2|^M$ (You may restrict your attention to one octant, where $\sigma_2 > \sigma_1 > 0$).

e. Bassani 2-D (
$$\sigma_3 = 0$$
): $f = |\sigma_1 + \sigma_2|^N + (1+2r) k |\sigma_1 - \sigma_2|^M$

SOLUTION:

a. von Mises:

$$\mathbf{f} = (\boldsymbol{\sigma}_1 - \boldsymbol{\sigma}_2)^2 + (\boldsymbol{\sigma}_1 - \boldsymbol{\sigma}_3)^2 + (\boldsymbol{\sigma}_2 - \boldsymbol{\sigma}_3)^2$$

$$\overline{\boldsymbol{\sigma}} = \frac{1}{\sqrt{2}} \left[\left(\boldsymbol{\sigma}_1 - \boldsymbol{\sigma}_2 \right)^2 + \left(\boldsymbol{\sigma}_1 - \boldsymbol{\sigma}_3 \right)^2 + \left(\boldsymbol{\sigma}_2 - \boldsymbol{\sigma}_3 \right)^2 \right]^{\frac{1}{2}}$$

 x_1 - Yield:

$$\frac{d\varepsilon_1}{2\sigma_1 - \sigma_2 - \sigma_3} = \frac{d\varepsilon_2}{2\sigma_2 - \sigma_1 - \sigma_3} = \frac{d\varepsilon_3}{2\sigma_3 - \sigma_1 - \sigma_2} \left(= \frac{d\varepsilon_i}{\sigma_i} \right)$$

Flow Rule:

Normality:

$$d\varepsilon_{1} = 2d\lambda \left(2\sigma_{1} - \sigma_{2} - \sigma_{3}\right)$$
$$d\varepsilon_{2} = 2d\lambda \left(2\sigma_{2} - \sigma_{1} - \sigma_{3}\right)$$

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$$d\varepsilon_{3} = 2d\lambda \left(2 \sigma_{3} - \sigma_{1} - \sigma_{2}\right)$$
Normality (inverse):

$$\sigma_{1} - \sigma_{2} = \frac{1}{6d\lambda} \left(d\varepsilon_{1} - d\varepsilon_{2}\right)$$

$$\sigma_{2} - \sigma_{3} = \frac{1}{6d\lambda} \left(d\varepsilon_{2} - d\varepsilon_{3}\right)$$

$$\sigma_{1} - \sigma_{3} = \frac{1}{6d\lambda} \left(d\varepsilon_{1} - d\varepsilon_{3}\right)$$
Lambda:

$$d\lambda = \frac{d\varepsilon}{4\sigma} \left(based \text{ on usage shown above}\right)$$
Effective Strain:

$$d\varepsilon = \frac{f_{2}^{2}}{3} \left[\left(d\varepsilon_{1} - d\varepsilon_{2}\right)^{2} + \left(d\varepsilon_{2} - d\varepsilon_{3}\right)^{2} + \left(d\varepsilon_{1} - d\varepsilon_{3}\right)^{2} \right]^{\frac{1}{2}}$$
Ratios $\left(\sigma_{3} = 0\right)$:

$$\alpha = \frac{\sigma_{2}}{\sigma_{1}} = \frac{2\beta + 1}{2 + \beta} \quad \beta = \frac{d\varepsilon_{2}}{d\varepsilon_{1}} = \frac{2\alpha - 1}{2 - \alpha}$$
Hill quadratic:

$$f = F \left(\sigma_{2} - \sigma_{3}\right)^{2} + G \left(\sigma_{1} - \sigma_{3}\right)^{2} + H \left(\sigma_{1} - \sigma_{2}\right)^{2}$$
Flow Rule:

$$\frac{d\varepsilon_{1}}{G \left(\sigma_{1} - \sigma_{3}\right) + H \left(\sigma_{1} - \sigma_{2}\right)} = \frac{d\varepsilon_{2}}{-F \left(\sigma_{2} - \sigma_{3}\right) - H \left(\sigma_{1} - \sigma_{2}\right)} = \frac{d\varepsilon_{3}}{-F \left(\sigma_{2} - \sigma_{3}\right) - G \left(\sigma_{1} - \sigma_{3}\right)}$$
Normality:

$$d\varepsilon_{1} = 2d\lambda \left[G \left(\sigma_{1} - \sigma_{3}\right) + H \left(\sigma_{1} - \sigma_{2}\right)\right]$$

Lambda

Effectiv

b. Hill qua

Ratios

$$\overline{\sigma} = \frac{1}{\sqrt{G+H}} \left[F \left(\sigma_2 - \sigma_3 \right)^2 + G \left(\sigma_1 - \sigma_3 \right)^2 + H \left(\sigma_1 - \sigma_2 \right)^2 \right]^{\frac{1}{2}}$$

Normali

$$d\varepsilon_{2} = 2d\lambda \left[F(\sigma_{2} - \sigma_{3}) - H(\sigma_{1} - \sigma_{2}) \right]$$
$$d\varepsilon_{3} = 2d\lambda \left[-F(\sigma_{2} - \sigma_{3}) - G(\sigma_{1} - \sigma_{3}) \right]$$
$$(\sigma_{1} - \sigma_{2}) = \frac{1}{2 \kappa d\lambda} \left(F d\varepsilon_{1} - G d\varepsilon_{2} \right)$$

Normality (inverse

$$\left(\sigma_{2} - \sigma_{3}\right) = \frac{1}{2 \kappa d\lambda} \left(G d\epsilon_{2} - H d\epsilon_{3}\right)$$

$$\left(\sigma_{1} - \sigma_{3}\right) = \frac{1}{2 \kappa d\lambda} \left(F d\epsilon_{1} - H d\epsilon_{3}\right)$$

$$\left(\text{where } \kappa = FG + FH + GH\right)$$

$$d\lambda = \frac{d\epsilon}{2 \left(G + H\right) \overline{\sigma}} \left(\text{based on usage shown above}\right)$$

Lambda:

Effective Strain:

$$d\bar{\epsilon} = \frac{\sqrt{G+H}}{FG+FH+GH} \left[F(Gd\epsilon_2 - Hd\epsilon_3)^2 + G(Fd\epsilon_1 - Hd\epsilon_3)^2 + H(Fd\epsilon_1 - Gd\epsilon_2)^2 \right]^{\frac{1}{2}}$$

Ratios
$$(\sigma_3=0)$$
:
 $\alpha = \frac{\sigma_2}{\sigma_1} = \frac{(G+H)\beta + H}{(F+H) + H\beta}$ $\beta = \frac{d\epsilon_2}{d\epsilon_1} = \frac{(F+H)\alpha - H}{(G+H) - H\alpha}$

$$\mathbf{f} = \left[\left(\boldsymbol{\sigma}_2 - \boldsymbol{\sigma}_3 \right)^2 + \left(\boldsymbol{\sigma}_1 - \boldsymbol{\sigma}_3 \right)^2 + \mathbf{r} \left(\boldsymbol{\sigma}_1 - \boldsymbol{\sigma}_2 \right)^2 \right]$$

c. Hill normal anisotropic:

$$\overline{\boldsymbol{\sigma}} = \frac{1}{\sqrt{1+r}} \left[\left(\boldsymbol{\sigma}_2 - \boldsymbol{\sigma}_3 \right)^2 + \left(\boldsymbol{\sigma}_1 - \boldsymbol{\sigma}_3 \right)^2 + r \left(\boldsymbol{\sigma}_1 - \boldsymbol{\sigma}_2 \right)^2 \right]^{\frac{1}{2}}$$

x₁ - Yield:

Flow Rule:
$$\frac{d\varepsilon_1}{\sigma_1 - \sigma_3 + r(\sigma_1 - \sigma_2)} = \frac{d\varepsilon_2}{\sigma_2 - \sigma_3 - r(\sigma_1 - \sigma_2)} = \frac{d\varepsilon_3}{2\sigma_3 - \sigma_1 - \sigma_2}$$

Normality:

$$d\varepsilon_{1} = d\lambda [\sigma_{1} - \sigma_{3} + r (\sigma_{1} - \sigma_{2})]$$

$$d\varepsilon_{2} = d\lambda [\sigma_{2} - \sigma_{3} - r (\sigma_{1} - \sigma_{2})]$$

$$d\varepsilon_{3} = d\lambda [2 \sigma_{3} - \sigma_{1} - \sigma_{2}]$$

$$(\sigma_{1} - \sigma_{2}) = \frac{1}{(1 + 2r) d\lambda} (d\varepsilon_{1} - d\varepsilon_{2})$$
Normality (inverse):

N

$$(\sigma_1 - \sigma_3) = \frac{1}{(1+2r) d\lambda} (d\epsilon_1 - r d\epsilon_3)$$
$$(\sigma_2 - \sigma_3) = \frac{1}{(1+2r) d\lambda} (d\epsilon_2 - r d\epsilon_3)$$
$d\bar{\epsilon} = \frac{1+r}{\sqrt{1+2r}} \left(d\epsilon_1^2 + d\epsilon_2^2 + \frac{2r}{1+r} d\epsilon_1 d\epsilon_2 \right)^{\frac{1}{2}}$

Lambda:

$$d\lambda = \frac{d\bar{\epsilon}}{(1+r)\,\bar{\sigma}}$$

Effective Strain:

Ratios
$$(\sigma_3=0)$$
: $\alpha = \frac{\sigma_2}{\sigma_1} = \frac{\beta(1+r)+1}{(1+r)+r\beta}$ $\beta = \frac{d\varepsilon_2}{d\varepsilon_1} = \frac{\alpha(1+r)-1}{(1+r)-r\alpha}$

d. Hill non-quadratic (Case IV):

$$f = (1+2r) (\sigma_1 - \sigma_2)^M + (\sigma_1 + \sigma_2)^M, \text{ (where } \sigma_2 > \sigma_2 > 0)$$

$$\overline{\boldsymbol{\sigma}} = \left\{ \frac{1}{2(1+r)} \left[(2r+1) (\boldsymbol{\sigma}_1 - \boldsymbol{\sigma}_2)^M + (\boldsymbol{\sigma}_1 + \boldsymbol{\sigma}_2)^M \right] \right\}^{\frac{1}{M}}$$

x₁ - Yield:

Flow Rule:

$$\frac{d\varepsilon_{1}}{(2r+1)(\sigma_{1}-\sigma_{2})^{M-1}+(\sigma_{1}+\sigma_{2})^{M-1}} = \frac{d\varepsilon_{2}}{-(2r+1)(\sigma_{1}-\sigma_{2})^{M-1}+(\sigma_{1}+\sigma_{2})^{M-1}} = \frac{d\varepsilon_{3}}{-2(\sigma_{1}+\sigma_{2})^{M-1}}$$
$$d\varepsilon_{1} = d\lambda \left[\frac{2r+1}{2(1+r)}(\sigma_{1}-\sigma_{2})^{M-1}+\frac{1}{2(1+r)}(\sigma_{1}+\sigma_{2})^{M-1}\right]$$
Normality:

$$d\varepsilon_{2} = d\lambda \left[\frac{-(2r+1)}{2(1+r)} (\sigma_{1} - \sigma_{2})^{M-1} + \frac{1}{2(1+r)} (\sigma_{1} + \sigma_{2})^{M-1} \right]$$
$$d\varepsilon_{3} = d\lambda \left[\frac{-1}{1+r} (\sigma_{1} + \sigma_{2})^{M-1} \right]$$

$$d\varepsilon_3 = d\lambda \left[\frac{1}{1+r} \left(\sigma_1 + \sigma_2 \right) \right]$$

$$\left(\sigma_{1}-\sigma_{2}\right) = \left[\frac{1}{d\lambda\left(1+2r\right)}\right]^{\frac{1}{M-1}} \left(d\varepsilon_{1}-d\varepsilon_{2}\right)^{\frac{1}{M-1}}$$

Normality (inverse):

$$(\sigma_1 + \sigma_2) = \left[\frac{1}{d\lambda}\right]^{\frac{1}{M-1}} (d\varepsilon_1 + d\varepsilon_2)^{\frac{1}{M-1}}$$
$$d\lambda = \frac{d\overline{\varepsilon}}{\overline{\sigma}^{M-1}}$$

Lambda:

$$d\boldsymbol{\tilde{\epsilon}} = \left[\frac{1}{2}\left(\frac{1+r}{1+2r}\right)^{\frac{1}{M-1}} \left(d\boldsymbol{\epsilon}_1 - d\boldsymbol{\epsilon}_2\right)^{\frac{M}{M-1}} + \frac{1}{2}\left(1+r\right)^{\frac{1}{M-1}} \left(d\boldsymbol{\epsilon}_1 + d\boldsymbol{\epsilon}_2\right)^{\frac{M}{M-1}}\right]^{\frac{M-1}{M}}$$

Effective Strain:

$$\alpha \ = \ \frac{\sigma_2}{\sigma_1} \ = \ \frac{\left(1 \ + \ 2r\right)\left(1 \ + \ \beta\right)^{\frac{1}{M-1}} - \left(1 \ - \ \beta\right)^{\frac{1}{M-1}}}{\left(1 \ + \ 2r\right)\left(1 \ + \ \beta\right)^{\frac{1}{M-1}} + \left(1 \ - \ \beta\right)^{\frac{1}{M-1}}}$$

Ratios ($\sigma_3 = 0$):

$$\beta = \frac{d\epsilon_2}{d\epsilon_1} = \frac{-(1+2r)(1-\alpha)^{M-1} + (1+\alpha)^{M-1}}{(1+2r)(1-\alpha)^{M-1} + (1+\alpha)^{M-1}}$$

e. Bassani yield function: $f = (\sigma_1 + \sigma_2)^N + k (\sigma_1 - \sigma_2)^M$ (where $\sigma_1 > \sigma_2 > 0$)

Virtually nothing can be done with this yield function in closed form. In fact, it is not dimensionally correct, because k must take the dimensionless form $k = \frac{N(2r+1)}{M}$ (Eq. 7.75), and thus the two terms in the yield function have different units and thus cannot even be added.

8. Verify that Figs. 7.21 represent the von Mises yield function when balanced biaxial tension and pure shear are used to define the effective stress.

SOLUTION:

$$\mathbf{k} = (\boldsymbol{\sigma}_1 - \boldsymbol{\sigma}_2)^2 + (\boldsymbol{\sigma}_1 - \boldsymbol{\sigma}_3)^2 + (\boldsymbol{\sigma}_2 - \boldsymbol{\sigma}_3)^2$$

a. Balanced biaxial tension: $\sigma_1 = \sigma_2 = \overline{\sigma}$, $\sigma_3 = 0$

$$k = \overline{\sigma}^2 + \overline{\sigma}^2 = 2\overline{\sigma}^2$$
 (same as Eq. 7.19)

b. Shear:

$$\mathbf{\sigma}_1 = -\mathbf{\sigma}_2 = \mathbf{\bar{\sigma}}, \quad \mathbf{\sigma}_3 = \mathbf{0}$$

$$\mathbf{k} = (2\mathbf{\bar{o}})^2 + \mathbf{\bar{o}}^2 + \mathbf{\bar{o}}^2 = 6\mathbf{\bar{o}}^2$$

9. Consider the strain paths A, B, and C shown in the figure.



Find the effective strain for each path using the following yield functions:

a.	von	Mises

b.	Hill - Orthotropic, Quadratic	$(r_{RD} = 2, r_{TD} = 1.5)$
с.	Hill - Normal Quadratic	(<i>r</i> = 1.75)
d.	Hill - Nonquadratic	(r = 1.75, M = 2.5)
е.	Hosford	(r = 1.75, M = 6)
f.	Bassani	(r = 1.75, N = 2, M = 4)

SOLUTION:

Path A:	Proportional from (0,0) to $(2,1) = \{\epsilon_1, \epsilon_2\}$
Path B:	B_1 proportional from (0, 0) to (2, 0), plus B_2 proportional from (2, 0) to (2, 1)

Path C: C_1 proportional from (0, 0) to (2, 2), plus C_2 proportional from (2, 2) to (2, 1)

a. von Mises:
$$d\tilde{\epsilon} = \left[\frac{2}{3}\left(d\epsilon_{1}^{2} + d\epsilon_{2}^{2} + d\epsilon_{3}^{2}\right)\right]^{\frac{1}{2}}$$
or, for a proportional path,

$$\Delta \bar{\epsilon} = \left[\frac{2}{3} \left(\Delta \epsilon_1^2 + \Delta \epsilon_2^2 + \Delta \epsilon_3^2\right)\right]^{\frac{1}{2}}$$
$$\bar{\epsilon}_A = \left\{\frac{2}{3} \left[2^2 + 1^2 + \left(-3\right)^2\right]\right\}^{\frac{1}{2}} = \boxed{3.06}$$

Path A:

B₁:
$$\Delta \tilde{\epsilon}_1 = \left\{ \frac{2}{3} \left[2^2 + 0^2 + (-2)^2 \right] \right\}^{\frac{1}{2}} = 2.31$$

Path B:

B₂:
$$\Delta \bar{\epsilon}_2 = \left\{ \frac{2}{3} \left[0^2 + 1^2 + (-1)^2 \right] \right\}^{\frac{1}{2}} = 1.15 \implies \overline{\epsilon_B} = 3.46$$

Path C:
Path C:

C₂:
$$\Delta \tilde{\epsilon}_2 = \left\{ \frac{2}{3} \left[0^2 + \left(-1 \right)^2 + \left(1 \right)^2 \right] \right\}^{\frac{1}{2}} = 115 \implies \overline{\epsilon_c = 5.15}$$

b. Hill orthotropic quadratic (see Eq. 7.56). For proportional paths and r_{RD} = 2, r_{TD} = 1.5 ,

$$\Delta \tilde{\epsilon} = 0.157 \left[2 \left(1.5 \Delta \epsilon_2 - 3 \Delta \epsilon_3 \right)^2 + 1.5 \left(2 \Delta \epsilon_1 - 3 \Delta \epsilon_3 \right)^2 + 3 \left(2 \Delta \epsilon_1 - 1.5 \Delta \epsilon_2 \right)^2 \right]^{\frac{1}{2}}$$

$$\tilde{\epsilon}_A = 0.157 \left[2 \left(10.5 \right)^2 + 1.5 \left(13 \right)^2 + 3 \left(2.5 \right)^2 \right]^{\frac{1}{2}} = \boxed{3.49}$$

A:

Path A:

Path B:
$$\Delta \tilde{\epsilon}_1 = 0.157 \left[2 \left(6 \right)^2 + 1.5 \left(10 \right)^2 + 3 \left(4 \right)^2 \right]^{\frac{1}{2}} = 2.58$$

$$\mathbf{B}_{2}: \quad \Delta \bar{\mathbf{\epsilon}}_{2} = 0.157 \left[2 \left(4.5 \right)^{2} + 1.5 \left(3 \right)^{2} + 3 \left(1.5 \right)^{2} \right]^{\frac{1}{2}} = 1.22 \implies \overline{\mathbf{\epsilon}_{\mathrm{B}}} = 3.80$$

Path C: $\Delta \tilde{\epsilon}_1 = 0.157 \left[2 (15)^2 + 1.5 (16)^2 + 3 (1)^2 \right]^{\frac{1}{2}} = 4.54$

C₂:
$$\Delta \tilde{\epsilon}_2 = 0.157 \left[2 \left(-4.5 \right)^2 + 1.5 \left(-3 \right)^2 + 3 \left(1.5 \right)^2 \right]^{\frac{1}{2}} = 1.22 \implies \overline{\epsilon_c = 5.76}$$

c. Hill normal quadratic (see Eq. 7.64) for proportional paths, $\dot{r} = 1.75$

$$\Delta \bar{\epsilon} = 1.30 \left[\Delta \epsilon_1^2 + \Delta \epsilon_2^2 + 1.27 \Delta \epsilon_1 \Delta \epsilon_2 \right]^{\frac{1}{2}}$$

A:
$$\bar{\epsilon}_A = 1.30 \left[2^2 + 1^2 + 1.27 \left(2 \right) \left(1 \right) \right]^{\frac{1}{2}} = \boxed{3.57}$$

Path A:

Path B:
$$\Delta \tilde{\epsilon}_1 = 1.30 \left[2^2 + 0^2 + 1.27 \left(2 \right) \left(0 \right) \right]^{\frac{1}{2}} = 2.60$$

B₂: Δε₂ = 1.30
$$\left[0^2 + 1^2 + 1.27 (0) (1)\right]^{\frac{1}{2}}$$
 = 1.3 ⇒ ε_B = 3.90

Path C:
$$\Delta \tilde{\epsilon}_1 = 1.30 \left[2^2 + 2^2 + 1.27 \left(2 \right) \left(2 \right) \right]^{\frac{1}{2}} = 4.70$$

C₂:
$$\tilde{\epsilon}_2 = 1.30 \left[0^2 + (-1)^2 + 1.27 (0) (-1) \right]^{\frac{1}{2}} = 1.30 \implies \overline{\epsilon_c} = 6.00$$

d. Hill non-quadratic (see Eq. 7.69), r = 1.75, M = 2.5:

$$\Delta \bar{\epsilon} = 0.99 \left[0.51 \left(\Delta \epsilon_1 - \Delta \epsilon_2 \right)^{1.667} + \left(\Delta \epsilon_1 + \Delta \epsilon_2 \right)^{1.667} \right]^{0.60}$$
l paths:

For proportional paths

Path A:

$$\bar{\epsilon}_{A} = 0.99 \left[0.51 \left(1 \right)^{1.667} + 3^{1.667} \right]^{0.60} = \overline{3.11}$$

Path B:
$$\Delta \tilde{\epsilon}_1 = 0.99 \left[0.51 \left(2 \right)^{1.667} + 2^{1.667} \right]^{0.60} = 2.54$$

B₂:
$$\Delta \tilde{\epsilon}_2 = 0.99 \left[0.51 \left| 2 \right|^{1.667} + 1^{1.667} \right]^{0.60} = 1.40 \implies \overline{\epsilon_B = 3.81}$$

Path C: $\Delta \tilde{\epsilon}_1 = 0.99 \left[0.51 \left(0 \right)^{1.667} + 4^{1.667} \right]^{0.60} = 3.96$

C₂:
$$\Delta \tilde{\epsilon}_2 = 0.99 \left[0.51 (1)^{1.667} + \left| -1 \right|^{1.667} \right]^{0.60} = 1.27 \implies \tilde{\epsilon}_C = 5.23$$

10. Assume that the tensile stress-strain curve for a standard material is well-known to be

$$\overline{\sigma} = 500 \ \overline{\epsilon}^{0.25}$$
 (MPa).

Consider a plane-strain tension test where $x_1 = RD = principal$ tensile axis, $x_2 = TD = zero$ strain direction, and $\sigma_3 = 0$.

- a. Use the yield functions in Problem 9 and find σ_l as a function of ε_l for this test.
- b. Plot the tensile stress-strain curves and σ_1 - ε_1 curves obtained in Part b on the same graph.

c. What is the stress ratio σ_2/σ_1 , for each plane-strain case?

SOLUTION:

Parts a and c - Strain ratios: ϵ , 0, $-\epsilon$ Stress ratios: $\sigma_1, \sigma_2, 0$

a. von Mises $\sigma_3 = 0$, $d\epsilon_2 = 0 \implies \sigma_2 = \frac{\sigma_1}{2}$ (see Eqs. 7.20 and 7.3-3)

$$\overline{\sigma} = \frac{1}{\sqrt{2}} \left[\left(\sigma_1 - \frac{\sigma_1}{2} \right)^2 + \sigma_1^2 + \left(\frac{\sigma_1}{2} \right)^2 \right]^{\frac{1}{2}}, \quad \overline{\sigma} = \frac{\sqrt{3}}{2} \sigma_1$$

$$\Delta \overline{\epsilon} = \left\{ \frac{2}{3} \left[d\epsilon_1^2 + 0^2 + \left(-d\epsilon_1 \right)^2 \right] \right\}^{\frac{1}{2}} = \frac{2}{\sqrt{3}} d\epsilon_1$$

$$\overline{\sigma} = 500 \ \overline{\epsilon}^{0.25} \implies \frac{\sqrt{3}}{2} \sigma_1 = 500 \left(\frac{2}{\sqrt{3}} \epsilon_1 \right)^{0.25}$$

$$\sigma_1 = \frac{2}{\sqrt{3}} 500 \left(\frac{2}{\sqrt{3}} \right)^{0.25} \epsilon_1^{0.25} = 598 \epsilon_1^{0.25}$$

b. Hill orthotropic quadratic (see Eqs. 7.54, 7.55b, and 7.56), $r_{RD} = 2$, $r_{TD} = 1.5$

$$\begin{split} \sigma_{3} &= 0, \ d\epsilon_{2} = 0 \implies r_{RD} \sigma_{2} - r_{RD} r_{TD} \left(\sigma_{1} - \sigma_{2}\right) = 0 \\ \left(-1 + r_{TD}\right) \sigma_{2} &= r_{TD} \sigma_{1} \\ \sigma_{2} &= \frac{r_{TD}}{1 + r_{TD}} \sigma_{1} = \frac{1.5}{2.5} \sigma_{1} = 0.60 \sigma_{1} \\ \bar{\sigma} &= \left(\frac{1}{1.5 (3)}\right)^{\frac{1}{2}} \left[2 \sigma_{2}{}^{2} + 1.5 \sigma_{1}{}^{2} + 3 (\sigma_{1} - \sigma_{2})^{2}\right]^{\frac{1}{2}} \\ &= (0.222)^{\frac{1}{2}} \left[2 (0.6)^{2} + 1.5 + 3 (0.4)^{2}\right]^{\frac{1}{2}} \sigma_{1} = 0.774 \sigma_{1} \\ \bar{\epsilon} &= 0.157 \left[2 (0 + 3 \epsilon_{1})^{2} + 1.5 (5 \epsilon_{1})^{2} + 3 (2 \epsilon_{1})^{2}\right]^{\frac{1}{2}} \\ \bar{\epsilon} &= 0.157 \left(18 + 37.5 + 12\right)^{\frac{1}{2}} \epsilon_{1} = 1.29 \epsilon_{1} \end{split}$$

So, 0.774
$$\sigma_1 = 500 (1.29 \epsilon_1)^{0.25}$$
, or $\sigma_1 = 687 \epsilon_1^{0.25}$

c. Hill normal quadratic (Eqs. 7.61b, 7.63, and 7.64), r = 1.75

$$d\varepsilon_{2} = 0, \quad \sigma_{2} = \frac{r}{1+r} \sigma_{1} = 0.64 \sigma_{1}$$

$$\overline{\sigma} = \left[1 + (0.64)^{2} - 1.28 (1) (0.64)\right]^{\frac{1}{2}} \sigma_{1} = 0.76 \sigma_{1}$$

$$\overline{\varepsilon} = (1.68)^{\frac{1}{2}} \left[1 + 0^{2} + 1.28 (0) (1)\right]^{\frac{1}{2}} \varepsilon_{1} = 1.30 \varepsilon_{1}$$
So, $0.76 \sigma_{1} = 500 (1.30 \varepsilon_{1})^{0.25} \Rightarrow \sigma_{1} = 694 \varepsilon_{1}^{0.25}$

d. Hill non-quadradic (see Eqs. 7.67, 7.68, and 7.69), r = 1.75, M = 2.5

$$d\varepsilon_{2} = 0, \quad \frac{2r+1}{2(1+r)} (\sigma_{1} - \sigma_{2})^{M-1} = \frac{1}{2(r+1)} (\sigma_{1} + \sigma_{2})^{M-1}$$

$$\underbrace{(2r+1)^{\frac{1}{M-1}}}_{k=2.72} (\sigma_{1} - \sigma_{2}) = (\sigma_{1} + \sigma_{2})$$

$$(k-1) \sigma_{1} = (k+1) \sigma_{2}, \text{ or } \sigma_{2} = \frac{k-1}{k+1} \sigma_{1} = 0.46 \sigma_{1}$$

$$\overline{\sigma} = \left[0.82 (0.53)^{2.5} + 0.18 (146)^{2.5}\right]^{\frac{1}{2.5}} \sigma_{1} = 0.83 \sigma_{1}, \qquad \overline{\varepsilon} = \left[0.51 + 1\right]^{0.60} \varepsilon_{1} = 1.32 \varepsilon_{1}$$
So, $0.83 \sigma_{1} = 500 (1.32 \varepsilon_{1})^{0.25} \Rightarrow \sigma_{1} = 646 \varepsilon_{1}^{0.25}$

Part b. - The plots are shown below.



11. Repeat Problem 9 for a balanced biaxial test where $\sigma_1 = \sigma_2$ and $\sigma_3 = 0$. Also find the strain ratios for each case.

a. von Mises (see Eqs. 7.20 and 7.3-3) isotropic, $\varepsilon_1 = \varepsilon_2$, $\varepsilon_3 = 2\varepsilon_1$

$$\begin{split} \mathbf{\bar{\sigma}} &= \frac{1}{\sqrt{2}} \left[\mathbf{\sigma}_{1}^{2} + \mathbf{\sigma}_{1}^{2} \right]^{\frac{1}{2}} = \mathbf{\sigma}_{1} \\ d\mathbf{\bar{\epsilon}} &= \left[\frac{2}{3} \left(\Delta \boldsymbol{\epsilon}_{1}^{2} + \Delta \boldsymbol{\epsilon}_{1}^{2} + 4 \Delta \boldsymbol{\epsilon}_{1}^{2} \right) \right]^{\frac{1}{2}} = 2\Delta \boldsymbol{\epsilon}_{1} \\ \mathbf{\bar{\sigma}} &= 500 \, \mathbf{\bar{\epsilon}}^{0.25} \implies \mathbf{\sigma}_{1} = 500 \left(2 \, \boldsymbol{\epsilon}_{1} \right)^{0.25} = 595 \, \boldsymbol{\epsilon}_{1}^{0.25} \end{split}$$

b. Hill orthotropic quadratic (see Eqs. 7.54, 7.55, and 7.56), $r_{RD} = 2$, $r_{TD} = 1.5$

$$\begin{split} \bar{\sigma} &= \left[\frac{1}{1.5\left(1+2\right)}\right]^{\frac{1}{2}} \left[2\,\sigma_{1}^{2}+1.5\,\sigma_{1}^{2}\right]^{\frac{1}{2}} = 0.88\,\sigma_{1} \\ \frac{d\,\varepsilon_{2}}{d\,\varepsilon_{1}} &= \frac{r_{RD}}{r_{TD}}, \text{ or } d\,\varepsilon_{2} = \frac{r_{RD}}{r_{TD}}\,d\,\varepsilon_{1} = \frac{4}{3}\,d\,\varepsilon_{1}\,d\,\varepsilon_{3} = -\frac{7}{3}\,d\,\varepsilon_{1} \\ \bar{\varepsilon} &= 0.157\left[2\left(1.5\,\frac{4}{3}\,\varepsilon_{1}+3\,\frac{7}{3}\,\varepsilon_{1}\right)^{2}+1.5\left(2\,d\,\varepsilon_{1}+3\,\frac{7}{3}\,\varepsilon_{1}\right)^{2}+3\left(2\,\varepsilon_{1}-1.5\,\frac{4}{3}\,\varepsilon_{1}\right)^{2}\right]^{\frac{1}{2}} \\ \bar{\varepsilon} &= 0.157\left[162+121.5+0\right]^{\frac{1}{2}}\varepsilon_{1} = 2.64\,\varepsilon_{1} \end{split}$$

$$\bar{\sigma} = 500 \,\bar{\epsilon}^{0.25} \implies 0.88 \,\sigma_1 = 500 \left(0.51 \,\epsilon_1\right)^{0.25}, \text{ or } \sigma_1 = 724 \,\epsilon_1^{0.25}$$

c. Hill normal quadratic (see Eqs. 7.63 and 7.64) r = 1.75

$$\begin{split} \bar{\sigma} &= \left[\frac{1}{2.75} \left(\sigma_{1}^{2} + \sigma_{1}^{2}\right)\right]^{\frac{1}{2}} = 0.85 \,\sigma_{1} \\ \frac{d \,\epsilon_{2}}{d \,\epsilon_{1}} &= \frac{1 - \frac{r}{1 + r}}{1 - \frac{r}{1 + r}} = 1 \\ \bar{\epsilon} &= \frac{2.75}{\sqrt{4.5}} \left(1 + 1 + \frac{3.5}{2.75} \,1\right)^{\frac{1}{2}} \epsilon_{1} = 2.35 \,\epsilon_{1} \\ \bar{\sigma} &= 500 \,\bar{\epsilon}^{0.25} \implies 0.85 \,\sigma_{1} = 500 \left(2.35 \,\epsilon_{1}\right)^{0.25}, \text{ or } \sigma_{1} = 728 \,\epsilon_{1}^{0.25} \end{split}$$

d. Hill non-quadratic (see Eqs. 7.67, 7.68, and 7.69), r = 1.75, M = 2.5

$$\begin{split} \overline{\sigma} &= 2 \left(\frac{1}{5.5} \right)^{\frac{1}{2.5}} \sigma_1 = 1.01 \sigma_1 \\ \frac{d \epsilon_1}{d \epsilon_2} &= \frac{\sigma_1^{-M-1}}{\sigma_1^{-M-1}} = 1 \\ \overline{\epsilon} &= 0.99 \left(2 \right) \epsilon_1 = 1.98 \epsilon_1 \qquad \overline{\sigma} = 500 \, \overline{\epsilon}^{0.25} \implies 1.01 \, \sigma_1 = 500 \left(1.98 \, \epsilon_1 \right)^{0.25} \\ \sigma_1 &= 587 \, \epsilon_1^{-0.25} \end{split}$$

12. Plot the 1st quadrant of the yield functions in Problem 9.

SOLUTION:

The plot below is in terms of normalized stresses, where the tensile flow stress in the x_1 direction is taken to be unity.



13. By plotting, show how r affects Hill's normal quadratic yield function. Take values of r = 1/2, r = 1 (von Mises), r = 2, r = 4 for illustration.



14. By plotting, show how M affects Hill's normal non-quadratic yield function. For r = 1, take values of M = 1.5, M = 2 (von Mises), M = 4, and M = 10.

SOLUTION:



15. Compare the form of Hill's normal non-quadratic yield function and Hosford's yield function by assuming that r = 2 and finding M in each case such that σ_1 (balanced biaxial tension) is equal to 1.2 $\overline{\sigma}_B$ (uniaxial tension).

For r = 2, σ_1 (balanced biaxial tension) = 1.1 σ

Hill normal non-quadratic (Eq. 7.67)

$$\overline{\sigma} = \left(\frac{1}{6} \cdot 2^{M}\right)^{\frac{1}{M}} \sigma_{1}$$
, therefore $1 = \left(\frac{1}{6}\right)^{M} \left(2\right) 1.1$, or $M = 2.272$ (Hill)

Hosford (Eq. 7.70)

$$\overline{\sigma} = \left(\frac{1}{3}\right)^{M} \left(2\right) \sigma_{1}$$
, therefore $1 = \left(\frac{1}{6}\right)^{M} \left(2\right) 1.1$, or $M = 1.39$ (Hosford)

The corresponding yield functions are thus

Hill:

$$\bar{\sigma}^{2.27} = \frac{5}{6} \left| \sigma_1 - \sigma_2 \right|^{2.27} + \frac{1}{6} \left| \sigma_1 + \sigma_2 \right|^{2.27}$$

$$\bar{\sigma}^{4.25} = \frac{1}{3} \left(\left| \sigma_1 \right|^{4.25} + \left| \sigma_2 \right|^{4.25} + 2 \left| \sigma_1 - \sigma_2 \right|^{4.25} \right)$$

Hosford:

The following plot of the first quadrant shows the differences of the two representations for the same r value and balanced-biaxial-to-tensile yield stress ratio.



B. DEPTH PROBLEMS

16. It has been proposed that the direction of the plastic strain increment, $d\mathbf{\epsilon}$, is not always normal to the yield surface. Instead, the proposer suggests that $d\hat{\mathbf{\varepsilon}}$ is intermediate in direction between $d\mathbf{\hat{\epsilon}}^{(n)}$. the normal direction, and d**σ**. the increment stress direction [i.e. $d\hat{\mathbf{\epsilon}} = \alpha d\hat{\mathbf{\epsilon}}^{(n)} + (1-\alpha) d\hat{\mathbf{\sigma}}, \text{ where } \alpha = 0 < \alpha < 1$ Criticize this model in terms of stability arguments and your knowledge of real materials.

SOLUTION:



The situation is as shown in the figure, where the choice of α , which might be a material property.

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Clearly, the choice of $d\boldsymbol{\varepsilon}$ for all α , between 0 and 1 satisfies second-order stability since $d\boldsymbol{\varepsilon}$ lies closer in direction to $d\hat{\boldsymbol{\sigma}}$ than does $d\hat{\boldsymbol{\varepsilon}}^{(n)}$. In general, the second-order work will be higher, for a given stress or strain path, which will be exhibited as higher work hardening.

The implication of the new direction of $d\mathbf{e}$ allows first order stability with some possible relaxation of convexity, depending on the choice of α .

17. It is convenient to introduce factors which can be used to multiply the effective stress or strain to obtain the stress or strain in a given state. For example:

$$\sigma_1^{BB} = F_{\sigma}^{(BB)} \overline{\sigma} \text{ or}$$
$$\epsilon_1^{BB} = F_{\epsilon}^{(BB)} \overline{\epsilon}$$

might be used to find the in-plane strains and stresses in balanced biaxial tension (BB) for a given yield function at a given hardness of $\overline{\epsilon}$, $\overline{\sigma}$.

- a. Show that F_{σ} and F_{ε} are constants with respect to strain for a given yield function when isotropic hardening is obeyed
- b. Find the specific values of $F_{\sigma}^{(PS)}$ and $F_{\varepsilon}^{(BB)}$ in terms of r which relate plane-strain tension to uniaxial tension using Hill's normal quadratic yield criterion.
- c. Repeat part b for Hill's normal nonquadratic yield function, finding F_{σ} and F_{ε} in terms of r and M.

SOLUTION:

The form of a yield function for an isotropic hardening material does not change as straining or hardening proceeds.

a. For a specified stress ratio ($^{\alpha}$) for a two dimensional state of stress,

$$\overline{\sigma} = \overline{\sigma} \left(\sigma_1, \sigma_2 \right) = \overline{\sigma} \left(\sigma_1, \alpha \sigma_1 \right) = f_{\sigma} \left(\alpha \right) \sigma_1, \text{ or } \frac{\sigma_1}{\overline{\sigma}} = F_{\sigma}$$

where $f(\alpha)$ and $F_{\sigma}(\alpha)$ are reciprocals fixed by the <u>form</u> of the yield function alone. (Note that for uniaxial tension, F_{σ} and f_{σ} are identically unity because $\overline{\sigma} = \sigma_1$. A similar argument follows for F_{ϵ} by noting that the plastic work is given by

$$\bar{\sigma} d\bar{\epsilon} = f_{\sigma}(\alpha) \sigma_{1} F_{\epsilon}(\beta) d\epsilon_{1} = f_{\sigma}(\alpha) F_{\epsilon}(\beta) \sigma_{1} d\epsilon_{1},$$

where f and F are constant functions, i.e. they do not depend on hardening.

$$0 = d\varepsilon_2 = \frac{d\tilde{\varepsilon}}{\bar{\sigma}} \left(\sigma_2 - \frac{r}{1+r} \sigma_1 \right), \text{ so}$$

$$\alpha = \frac{\sigma_2}{\sigma_1} = \frac{r}{1+r}, \ \sigma_2 = \alpha \sigma_1, \ \sigma_2 = \frac{r}{1+r} \sigma_1$$

Then we can find $F_{\sigma}^{(PS)}$ from Eq. 7.63,

$$\overline{\sigma}^{2} = \left[\sigma_{1}^{2} + \left(\frac{r}{1+r}\right)^{2}\sigma_{1}^{2} - \frac{2r}{1+r}\left(\frac{r}{1+r}\right)\sigma_{1}^{2}\right]$$
$$\overline{\sigma} = \frac{\sqrt{1+2r}}{1+r}\sigma_{1}, \quad F_{\sigma}^{(PS)} = \frac{1+r}{\sqrt{1+2r}}$$

And, similarly for $F_{\epsilon}^{(PS)}$ from Eq. 7.64:

$$d \tilde{\epsilon}^{2} = \frac{\left(1+r\right)^{2}}{1+2r} \left(d \epsilon_{1}^{2} + 0^{2} + \frac{2r}{1+r} d \epsilon_{1} \left(0\right) \right)^{2}$$
$$d \tilde{\epsilon} = \frac{1+r}{\sqrt{1+2r}} d \epsilon_{1}, \ F_{\epsilon}^{(PS)} = \frac{\sqrt{1+2r}}{1+r}$$

The procedure is identical for balanced biaxial tension, where $\alpha = 1$ and $d \epsilon_1 = d \epsilon_2$ (from Eq. 7.63a and b):

$$\begin{split} \overline{\boldsymbol{\sigma}} &= \left[1 + 1 - \frac{2r}{1+r} \left(l\right) \left(l\right)\right] \boldsymbol{\sigma}_{1}, \qquad \overline{\boldsymbol{\sigma}} = \sqrt{\frac{2}{1+r}} \, \boldsymbol{\sigma}_{1}, \ F_{\boldsymbol{\sigma}}^{(BB)} = \sqrt{\frac{1+r}{2}} \\ d \, \tilde{\boldsymbol{\epsilon}} &= \frac{1+r}{\sqrt{1+2r}} \left[2 + \frac{2r}{1+r}\right]^{\frac{1}{2}} d \, \boldsymbol{\epsilon}_{1}, \qquad d \, \tilde{\boldsymbol{\epsilon}} = \sqrt{2\left(l+r\right)} \, d \, \boldsymbol{\epsilon}_{1}, \ F_{\boldsymbol{\epsilon}}^{(BB)} = \frac{1}{\sqrt{2\left(l+r\right)}} \end{split}$$

c. The solution fowllows Part b with the use of Eqs. 7.67-7.69.

For plane-strain tension:

$$d \epsilon_{2} = 0 \implies \alpha = \frac{\sigma_{2}}{\sigma_{1}} = \frac{(2r+1)^{\frac{1}{M-1}} - 1}{(2r+1)^{\frac{1}{M-1}} + 1} = \frac{\kappa-1}{\kappa+1}, \text{ where: } \kappa \equiv (2r+1)^{\frac{1}{M-1}}$$

$$F_{\sigma}^{(PS)} = \left[\frac{\left(1+\kappa\right)\left(1+r\right)^{\frac{1}{M-1}}}{2\kappa}\right]^{\frac{M-1}{M}}, \qquad F_{\epsilon}^{(PS)} = \left[\frac{2\kappa}{\left(1+\kappa\right)\left(1+r\right)^{\frac{1}{M-1}}}\right]^{\frac{M-1}{M}},$$

For balanced biaxial tension $\left(\sigma_1 = \sigma_2, d \varepsilon_1 = d \varepsilon_2\right)$:

$$F_{\sigma}^{\ (PS)} \ = \ \frac{\left[2\left(1+r\right) \right]^{\frac{1}{M}}}{2} \ , \qquad \qquad F_{\epsilon}^{\ (PS)} \ = \left[\frac{1}{2\left(1+r\right)} \right]^{\frac{1}{M}}$$

d. Inspection shows that in plane strain tension:

$$F_{\sigma}^{\ (PS)} \ = \ \frac{1}{F_{\epsilon}^{\ (PS)}} \ . \label{eq:Fs}$$

This result follows the definition of effective strain from the principle of equivalent plastic work:

$$\label{eq:states} \boldsymbol{\bar{\sigma}}\,d\,\boldsymbol{\bar{\epsilon}} \;=\; \boldsymbol{\sigma}_1\,d\,\boldsymbol{\epsilon}_1 + \boldsymbol{\sigma}_2\,d\,\boldsymbol{\epsilon}_2 + \boldsymbol{\sigma}_3\,d\,\boldsymbol{\epsilon}_3.$$

For plane-strain tension, $\sigma_3 = 0$, d $\varepsilon_2 = 0$, so,

 $\bar{\sigma} d\bar{\epsilon} = \sigma_1 d\epsilon_1 = F_{\sigma} \bar{\sigma} F_{\epsilon} d\bar{\epsilon}, \text{ so } F_{\sigma} F_{\epsilon} \equiv 1$

- 18. In view of results from Problem 17, what can you say about the complexity of yield function which would be required to account for the following observations? (Assume only normal anisotropy.)
 - a. In uniaxial tension, $\sigma_I = 500 \ \varepsilon_I^{0.25}$ In plane-strain tension, $\sigma_I = 600 \ \varepsilon_I^{0.25}$
 - b. Same as part a, but r = 2 from the tensile test.
 - c. In uniaxial tension, $\sigma_I = 500 \ \varepsilon_I^{0.25}$ In plane-strain tension, $\sigma_I = 600 \ \varepsilon_I^{0.35}$ r = 2.

SOLUTION:

- a. The difference in strength coefficient can be accounted for by a single value of r, so Hill's normal quadratic theory is sufficient.
- b. If r is known independently, then we need a second adjustable parameter, M, to fit the data.
- c. None of the standard, isotropic-hardening theories can account for different hardening rates. See, however, R. H. Wagoner: <u>Metall. Trans. A</u>., 1980, vol. 11A, pp. 165-175 for a modification of standard theories to account for such behavior.
- 19. Various authors have attempted to introduce work hardening parameters, particularly to compare

various strain states or to compare material hardness when true strain is unknown. Here are three such quantities:

$d\bar{\sigma}$	$d\overline{\sigma}$ $d \ln \overline{\sigma}$	$\varepsilon d\sigma d \ln \sigma$
dε	$\sigma d\epsilon = d\epsilon$	$\overline{\sigma} d\varepsilon = \overline{d \ln \varepsilon}$

Based on your knowledge of isotropic hardening and results from Problems 17 and 18, which of these do you think is most suitable for comparing hardening in various strain states?

SOLUTION:

Remember that effective quantities, σ and ε , depend on the choice of yield function. For a given set of $\sigma_1 - \varepsilon_1$ data (from a plane-strain test, for example), the corresponding $\overline{\sigma} - \varepsilon$ will be obtained after assumption of a yield function, and the corresponding quantities F_{σ} and F_{ε} :

$$\frac{d\,\overline{\sigma}}{d\,\overline{\epsilon}} = \frac{F_{\epsilon}\,d\,\sigma_{1}}{F_{\sigma}\,d\,\epsilon_{1}}$$

$$\frac{d\,\ln\overline{\sigma}}{d\,\overline{\epsilon}} = \frac{d\,\overline{\sigma}}{\overline{\sigma}\,d\,\overline{\epsilon}} = \frac{F_{\epsilon}\,d\,\sigma_{1}}{d\,\epsilon_{1}} \left(\text{the } \frac{d\,\overline{\sigma}}{\overline{\sigma}} \text{ remains } F_{\sigma} \right)$$

$$\frac{d\,\ln\overline{\sigma}}{d\,\overline{\epsilon}} = \frac{\tilde{\epsilon}\,d\,\overline{\sigma}}{\overline{\sigma}\,d\,\overline{\epsilon}} = \frac{\epsilon_{1}\,d\,\sigma_{1}}{\sigma_{1}\,d\,\epsilon_{1}} = \frac{d\,\ln\sigma_{1}}{d\,\ln\epsilon_{1}}$$

Therefore, the third measure of work hardening is invariant to the choice of yield function, contrary to the other two.

See R. H. Wagoner: <u>Metall. Trans. A</u>, 1981, vol. 12A, pp. 2142-2145 for a more thorough discussion of work-hardening measures.

CHAPTER 8 - PROBLEM SOLUTIONS

A. PROFICIENCY PROBLEMS

1. In an effort to refine the upper bound estimate to density, ρ , that a vertical embankment of height h can sustain, you propose a trial displacement field that is different than shown in Exercise 8.1. The field is shown below, where the boundary between the slipping and stationary parts of the embankment is a quarter circle of radius R, from $\theta = 0$ to $\pi/2$. Calculate the new upper bound to the density.



SOLUTION:

Take $d\delta$, the relative displacement across the boundary, to equal Rd ϕ , where $d\phi$ is the incremental rotation of the quarter circle of material about the center. The internal work per unit depth of embankment is given by

$$IW = \frac{2\pi R}{4} kRd\phi \ .$$

The external work is given by considering a piece of material with differential area, r dr d θ , on which a downward gravitational force, $f_g = \rho gr dr d\theta$, acts per unit depth of embankment. The work increment per unit area of material is then $f_g r \cos\theta d\delta$, where the factor $\cos\theta$ is required to project \mathbf{f}_g onto the same direction as $d\mathbf{\delta}$. The external work is then

$$EW = \rho g d\phi \int_{0}^{R \pi/2} \int_{0}^{r^2} \cos \theta dr d\theta = \frac{\rho g R^3 d\phi}{3} .$$

Equating IW to EW produces the following upper bound,

$$\rho \leq \frac{3\pi k}{2\,\mathrm{gR}}$$

The minimum upper bound on ρ is given by making R = h, the largest value possible. For this assumed displacement field,

$$\rho \le \rho_{\min} = \frac{3\pi k}{2\,\mathrm{gh}} \; .$$

The "quarter circle" deformation mode produces a better upper bound than the linear shearing off mechanism presented in Exercise 8.1.

2. Consider the proposed deformation field shown in Fig. 8.3(a). Discuss why for general θ_1 , θ_2 the stress state at the lower tip of triangular block d can not be an equilibrium stress state.

Below is a sketch of the lower tip of triangular block d. One can not construct a Mohr's circle for the stress state at this point. In particular, if you rotate couterclockwise by $180-\theta_1-\theta_2$, the shear stress must change from a maximum of k on the plane 1 to -k on plane 2. This is not possible for general θ_1 , θ_2 .



3. Construct a lower bound for the indentation load, P, per unit depth, based on a three-sector stress field. Vary the angle θ to obtain the best lower bound.



SOLUTION:

The optimal lower bound is given by setting $\theta = 90^\circ$, so that the Mohr's circle construction for the two sectors, A and B, is two circles of radius k as shown below.



The best lower bound here is P = 4kw.

4. Using the yield condition, Eq. 8.13, show why hydrostatic loading, $\sigma_{11} = \sigma_{22} = \sigma_{33} = \sigma$, cannot cause yield on any slip system, regardless of the orientation of the slip system to the loading axes.

If the stress state, $\sigma_{11} = \sigma_{22} = \sigma_{33} = \sigma$ is substituted into the yield condition, Eq. 8.13, then $\pm \left(s_1^{(\alpha)} m_1^{(\alpha)} + s_2^{(\alpha)} m_2^{(\alpha)} + s_3^{(\alpha)} m_3^{(\alpha)} \right) \ge \tau_c^{(\alpha)}$

or equivalently, $\pm (s^{(\alpha)} \cdot \underline{m}^{(\alpha)}) \sigma \ge \tau_c^{(\alpha)}$

For slip planes, <u>s</u> and <u>m</u> must be perpendicular to one another, so that $\underline{s} \cdot \underline{m} = 0$ for all slip systems. This statement and the rephrased yield condition above predict that yield can never be reached under hydrostatic loading.

5. Experiments which apply hydrostatic loading to engineering materials have shown that materials such as steel, aluminum, copper, or silicon will yield, although the magnitude of hydrostatic loading to cause yield is many times that required for simple tension or compression. Discuss why materials do yield in hydrostatic loading.

SOLUTION:

Typically, engineering grades of materials contain grain boundaries, inclusions, and voids. All serve to act as stress concentrators, so that under nominal hydrostatic loading, the stress state is not homogeneous and entirely hydrostatic. Yield can then occur.

- 6. A single crystal is indented with a square knife edge as shown below. The candidate slip systems are oriented at discrete angles θ_1 , θ_2 , and θ_3 .
 - a. Construct a deformation field out of rigid, sliding triangles, produce the corresponding hodograph, and determine an upper bound to the indentation load.
 - b. If the crystal had only two slip planes, could you construct a deformation field? Support your answer with some sketches. What would the indentation load P be under such a case?



SOLUTION:

a. The deformation field and the corresponding hodograph are:



The external dissipation rate is Pv. The internal dissipation rate is $k[l_{BC'}V_{BC'} + l_{C'O}V_{C'O} + l_{C'D}V_{C'D} + l_{D'O}V_{D'O} + l_{BC}V_{BC} + l_{CO}V_{CO} + l_{CD}V_{CD} + l_{DO}V_{DO}].$

Substituting the values for l_{IJ} derived from the deformation diagram and values for v_{IJ} derived from the hodograph, the internal dissipation rate is approximately 6.93 kwv. Equating the external and internal dissipation rates gives: P = 6.93 kw.

- b. It is not possible to construct a deformation field to this geometry with only two slip directions, provided that the sample is infinitely thick. Triangular blocks must be constructed for a successful deformation field, and the three sides of the block will require three slip directions.
- 7. In the upper bound analysis of a block of dimensions l_1 , l_2 , and l_3 , which is sheared by an amount b along a direction <u>s</u>, on a plane with normal <u>m</u> (see Fig. 8.2), the internal dissipation is $IW^* = k Ab$

and the external dissipation is

 $EW^* = \sigma_{11}^* Abm_1s_1 + \sigma_{22}^* Abm_2s_2$

where σ_{11}^* and σ_{22}^* are uniform stresses, and A is the area of slipped plane.

- a. Sketch the resulting upper bound load to the collapse surfaces in $\sigma_{11}^*-\sigma_{22}^*$ space, assuming the inclination angle, θ , shown in Fig. 8.2 equals 30°.
- b. Although the internal strain is concentrated at the slip plane, define an appropriate average, or macroscopic strain state for the block, after a slip of b has occurred on the plane.
- c. Do your macroscopic strains in Part b satisfy normality with the collapse surface in Part a? Explain why. If they do not propose a definition of macroscopic strain that will satisfy normality.

SOLUTION:

a. We note that $k = \sigma_{11}*m_1s_1 + \sigma_{22}*m_2s_2$, where for this problem, $m_1 = \cos 60$, $s_1 = +\cos 30$, $m_2 = \cos 30$, and $s_2 = -\cos 60$. When values of m_i and s_i are substituted into the expression,

$$(\sigma_{11}^* - \sigma_{22}^*) = \pm \frac{4}{\sqrt{3}} k$$

where the \pm sign on the RHS is produced by considering positive or negative values of slip on the plane. The result is shown below.



b. Using Eq. 8.23, $d\bar{\epsilon}_{ij} = \frac{1}{2} \frac{Ab}{l_1 l_2 l_3} (m_j s_i + m_i s_j)$

so that using the particular values for the m_i, s_i,

$$d\bar{\varepsilon}_{11} = \pm \frac{\sqrt{3}}{4} \frac{Ab}{l_1 l_2 l_3}$$
$$d\bar{\varepsilon}_{22} = \mp \frac{\sqrt{3}}{4} \frac{Ab}{l_1 l_2 l_3}$$

- c. The macroscopic strains defined in part (b) do satisfy normality. The strain increments may be plotted in the $\sigma_{11}^* \sigma_{22}^*$ diagram in (a), where the ε_{11}^* axis is constructed parallel to the σ_{11}^* axis, and the ε_{22}^* axis is constructed parallel to the σ_{22}^* axis. The incremental strains are then at +135°, -45° angles measured counterclockwise from the horizontal axis. They are perpendicular to the yield surface.
- 8. Construct the projection of the f.c.c. yield surface on to the σ_{11} - σ_{12} stress plane. Compare your answer to the prediction based on a Tresca yield criterion, $\sigma_1 \sigma_3 = \pm 2k$, where σ_1 and σ_3 are the maximum and minimum principal stresses, respectively.

SOLUTION:

We note Eq. 8.17 and pick $\sigma_1 = \sigma_{11}$ and $\sigma_6 = \sigma_{12} (=\sigma_{21})$ to be the only non-zero components of stress. The resulting yield condition becomes

$$\begin{bmatrix} \sigma_{11} \\ \sigma_{11} + \sigma_{12} \\ \sigma_{12} \\ -\sigma_{11} + \sigma_{12} \\ -\sigma_{12} \\ -\sigma_{12} \\ \sigma_{11} \\ -\sigma_{11} \\ \sigma_{11} \\ \sigma_{11} \\ \sigma_{11} \\ \sigma_{11} \\ \sigma_{12} \\ \sigma_{12} \\ \sigma_{12} \end{bmatrix}$$

Here, the assumption is made that the critical resolved shear stress for each plane is k. The corresponding yield surface is constructed from lines generated from the twelve slip systems represented in the above equation.

The corresponding Tresca yield condition is generated from $|\sigma_1 - \sigma_3| = 2k$. This condition states that the diameter, D, of Mohr's circle, constructed in σ_{11} - σ_{12} space, can not exceed 2k. The diameter of Mohr's circle for this stress state is

$$D = 2\sigma_{12}^2 + \frac{1}{2}\sigma_{11}^2 (= 2k)$$

Accordingly, the above is an equation of an ellipse for which the σ_{11} intercept is $\pm 2k$ and the σ_{12} intercept is $\pm k$. It is shown in the figure below.



9. Imagine that you can load a f.c.c. crystal along any crystallographic direction. Assume that the potential slip systems are of the type $\{111\}/\langle110\rangle$, as listed in Table 8.1. Find a crystallographic direction along which the tensile stress to yield is a minimum, and report the minimum value of tensile stress in terms of τ_c , the critical resolved shear stress to activate slip.

Use the yield condition, Eq. 8.13, expressed in the form, $\tau^{(\alpha)} = (s^{(\alpha)} \cdot e^{(1')})(m^{(\alpha)} \cdot e^{(1')})\sigma \ge \pm \tau_c^{(\alpha)} \quad ,$

where $\mathbf{s}^{(\alpha)}$ and $\mathbf{m}^{(\alpha)}$ are the slip direction and slip plane normal of slip system (α), and $\mathbf{e}^{(1')}$ is the direction along which the tensile stress, σ , is applied. The minimum σ is found by making the factor involving the dot products as large as possible. The largest dot products are obtained by first choosing $\mathbf{e}^{(1')}$ to lie in the plane containing $\mathbf{s}^{(\alpha)}$ and $\mathbf{m}^{(\alpha)}$. Since $\mathbf{s}^{(\alpha)}$ and $\mathbf{m}^{(\alpha)}$ are orthogonal to each other, the relative orientations of vectors is depicted below, where $\mathbf{s}^{(\alpha)} \cdot \mathbf{e}^{(1')} = \sin\theta$ and $\mathbf{m}^{(\alpha)} \cdot \mathbf{e}^{(1')} = \cos\theta$. Therefore,

$$\sigma = \frac{\tau_c}{|\sin\theta\cos\theta|} \quad ,$$

and the minimum value, $\sigma = 2\tau_c$, occurs when $\theta = 45^\circ$, 135°, 225°, or 315°. There are several possible crystallographic directions of the type $\mathbf{e}^{(1')} = \pm (\mathbf{s}^{(\alpha)} \pm \mathbf{m}^{(\alpha)})/\sqrt{2}$. A possible direction for the {111}/<110> slip systems considered here are $[\sqrt{2} + \sqrt{3} \sqrt{2} - \sqrt{3} \sqrt{2}]/2\sqrt{3}$.

10. A biaxial stress state is applied to a single crystal of f.c.c. material. However, the crystal is oriented so that one applied stress, $\sigma_{1'1'}$, is along the [111] crystallographic direction, and the other applied stress, $\sigma_{2'2'}$, is along the [1-1 0] crystallographic direction. Assume that all slip planes of the type {111}/<110> (see Table 8.1) slip when the critical resolved shear stress reaches τ_c . Construct a projection of the yield surface onto the $\sigma_{1'1'}$ - $\sigma_{2'2'}$ stress plane. Comment on the difference between this projection and the one shown in Fig. 8.4.

SOLUTION:

A relatively simple approach is to revert to the yield condition, Eq. 8.13, and compute directly the resolved shear stress on each of the twelve slip systems according to

$$\tau^{(\alpha)} = (\mathbf{s}^{(\alpha)} \cdot \mathbf{e}^{(1')})(\mathbf{m}^{(\alpha)} \cdot \mathbf{e}^{(1')})\sigma_{1'1'} + (\mathbf{s}^{(\alpha)} \cdot \mathbf{e}^{(2')})(\mathbf{m}^{(\alpha)} \cdot \mathbf{e}^{(2')})\sigma_{2'2'} \ge \pm \tau_c^{(\alpha)} ,$$

where $\mathbf{e}^{(1')}$ and $\mathbf{e}^{(2')}$ corresponds to the crystallographic directions $[111]/\sqrt{3}$ and $[1-1\ 0]/\sqrt{2}$, and the vectors $\mathbf{s}^{(\alpha)}$ and $\mathbf{m}^{(\alpha)}$ correspond to the slip directions and slip plane normals, respectively, of slip system (α). The dot products involved project the components of stress, applied along the 1' and 2' directions, on to the $\mathbf{s}^{(\alpha)}$ and $\mathbf{m}^{(\alpha)}$ directions. The twelve slip systems listed in Table 8.1 are used in the yield condition and produce twelve equations of the form,

$$\frac{\sigma_{2'2'}}{\sqrt{6}} \ge \pm \tau_c \quad \text{(slip systems 7, 5)}$$

$$\frac{2\sigma_{1'1'}}{3\sqrt{6}} \ge \pm \tau_c \quad \text{(slip systems 6, 9, 11, 12)}$$

$$\frac{2\sigma_{1'1'}}{3\sqrt{6}} + \frac{\sigma_{2'2'}}{\sqrt{6}} \ge \pm \tau_c \quad \text{(slip systems 4, 8)}$$

The result indicates that two or more slip systems can produce the same yield condition. Slip systems 1, 2, 3, and 10 have zero resolved shear stress on them, and hence, will never yield.

The yield surface is depicted by the dotted line in the figure below. For comparison, the

projection of the yield surface on to the σ_{11} - σ_{22} plane is reproduced from Fig. 8.4 and is indicated by the solid line. The differences in yield surfaces demonstrates that yield behavior depends on the choice of loading axes relative to the crystal basis.



A more involved approach is to use Eq. 8.17. The components, σ_i , of stress in that equation refer to the cube basis of the f.c.c. crystal. However, the applied stress is expressed in terms of components along the [111] and [1-1 0] directions. Thus, Eq. 8.17 may be used only if the applied stress state, $\sigma_{1'1'}$ and $\sigma_{2'2'}$, is expressed in terms of the components in the crystal basis. Let the basis $e^{(1)}$, $e^{(2)}$, and $e^{(3)}$ correspond to the crystallographic directions, [100], [010], and [001], and the basis $e^{(1')}$ and $e^{(2')}$ corresponds to the crystallographic directions [111]/ $\sqrt{3}$ and [1-1 0]/ $\sqrt{2}$, as before. Then

$$\sigma_{ij} = A_{ik}A_{jl}\sigma'_{kl}$$

where $A_{ik} = e^{(i)} \cdot e^{(k')}$, σ_{ij} corresponds to components in the $e^{(i)}$ basis, and σ'_{ij} corresponds to components in the $e^{(i')}$ basis. Accordingly,

	1/√3	1/√2	*	
[A _{ik'}]=	1/√3	$-1/\sqrt{2}$	*	,
	1/√3	0	*	

and the computed components of stress are



Substitution into Eq. 8.17 produces the same three types of yield conditions derived from the first approach.

11. A sphere with radius r = 1 cm is a single f.c.c. crystal. Slip systems 1, 6, and 12 listed in Table 8.1

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are activated, so that a slip of $1\mu m$ in the corresponding +s directions listed occurs on slip planes which pass right through the center of the sphere. Calculate the resulting macroscopic strain in the crystal.

SOLUTION:

The resulting macroscopic strain is given by

$$\bar{\varepsilon}_{ij} = \sum_{\alpha} \frac{1}{2} \frac{A_s^{(\alpha)} b^{(\alpha)}}{V} \left(m_i^{(\alpha)} s_j^{(\alpha)} + m_j^{(\alpha)} s_i^{(\alpha)} \right)$$

where the sum α is over slip systems 1, 6, and 12. The slip area $A_s = \pi r^2$ and the amount of slip $b^{(\alpha)} = 1 \mu m$ for the three slip systems. Also, $V = 4\pi r^3/3$, where r = 1 cm. Consequently, the equation for average strain above yields contributions for each of the three active slip systems,

$$\bar{\varepsilon}_{ij} = \frac{3}{8\sqrt{6}} \cdot 10^{-4} \left[\left(m_j s_i + m_i s_j \right)^{(1)} + \left(m_j s_i + m_i s_j \right)^{(6)} + \left(m_j s_i + m_i s_j \right)^{(12)} \right]$$

Substituting values of $m_{j} \text{ and } s_{i}$ for each of the slip systems,

$$\begin{bmatrix} \overline{\mathbf{\epsilon}}_{ij} \end{bmatrix} = \frac{3}{8\sqrt{6}} \cdot 10^{-4} \begin{bmatrix} 0 & 1 & 3 \\ 1 & 2 & 0 \\ 3 & 0 & -2 \end{bmatrix}$$

12. Consider the portion of the yield surface in Fig. 8.4 that is contributed by slip systems 2, 4, 7, and 11. Using Eq. 8.25, show that operation of any individual or linear combination of these slip systems produces a strain increment which is perpendicular to that portion of the yield surface shown in the figure.

SOLUTION:

The relevant equation is
$$\bar{\epsilon}_{ij} = \sum_{\alpha} \frac{d\gamma^{(\alpha)}}{2} \left(m_i s_j + m_j s_i \right)^{\alpha}$$

Applying this to slip systems 2 and 11, we find $d\overline{\epsilon}_{11}^{(\alpha)} = \frac{d\gamma^{(\alpha)}}{\sqrt{6}}$, $d\overline{\epsilon}_{22}^{(\alpha)} = 0$

and applying this to slip systems 4 and 7, $d\overline{\epsilon}_{11}^{(\alpha)} = -\frac{d\gamma^{(\alpha)}}{\sqrt{6}}$, $d\overline{\epsilon}_{22}^{(\alpha)} = 0$

Since each individual system produces zero $d\epsilon_{22}$, any linear combination of operation of these slip systems will do the same. When plotted in ϵ_{11} - ϵ_{22} space, the strain increment is perpendicular to the yield surface lines contributed by slip systems 2, 4, 7, and 11 in Fig. 8.4. Note that components of strain other than ϵ_{11} may be non-zero here, but that the projected strain increment in ϵ_{11} - ϵ_{22} space is still normal to the yield surface in σ_{11} - σ_{22} space shown.

13. Equation 8.24 states that external and internal plastic work increments are equal. Show that this equation may be converted to the notation, $\sigma_i d\varepsilon_i = (\tau_c d\gamma)^{(\alpha)}$, where the components $[\sigma_i] = [\sigma_1, \sigma_2, \sigma_3, \sigma_4, \sigma_5, \sigma_6] = [\sigma_{11}, \sigma_{22}, \sigma_{33}, \sigma_{23}, \sigma_{13}, \sigma_{12}]$, and the components $[d\varepsilon_i]$ are defined by the identity, Eq. 8.31. **SOLUTION:**

Equating internal and external plastic work increments produces $\sigma_{ij}d\epsilon_{ij} = \sum_{(\alpha)} \tau_c^{(\alpha)} d\gamma^{(\alpha)}$

or if the contracted notation σ_i , $d\epsilon_i$ is used, then the following must hold for each slip system (α) $\sigma_i d\epsilon_i^{(\alpha)} = \tau_c^{(\alpha)} d\gamma^{(\alpha)}$.

Equilibrium requires that the stress state produces a resolved shear stress on each active slip system such that

$$\begin{aligned} \tau_{c}^{(\alpha)} &= \sigma_{1} (m_{1}s_{1})^{(\alpha)} + \sigma_{2} (m_{2}s_{2})^{(\alpha)} + \sigma_{3} (m_{3}s_{3})^{(\alpha)} \\ &+ \sigma_{4} (m_{2}s_{3} + m_{3}s_{2})^{(\alpha)} \\ &+ \sigma_{5} (m_{1}s_{3} + m_{3}s_{1})^{(\alpha)} \\ &+ \sigma_{6} (m_{1}s_{2} + m_{2}s_{1})^{(\alpha)} \end{aligned}$$

When this is used to replace $\tau_c^{(\alpha)}$ in Eq. (*) above, then there equivalence of work only if

$$d\varepsilon_{1}^{(\alpha)} = (m_{1}s_{1})^{(\alpha)}d\gamma^{(\alpha)}$$

$$d\varepsilon_{2}^{(\alpha)} = (m_{2}s_{2})^{(\alpha)}d\gamma^{(\alpha)}$$

$$d\varepsilon_{3}^{(\alpha)} = (m_{3}s_{3})^{(\alpha)}d\gamma^{(\alpha)}$$

$$d\varepsilon_{4}^{(\alpha)} = (m_{2}s_{3} + m_{3}s_{2})^{(\alpha)}d\gamma^{(\alpha)}$$

$$d\varepsilon_{5}^{(\alpha)} = (m_{1}s_{3} + m_{3}s_{1})^{(\alpha)}d\gamma^{(\alpha)}$$

$$d\varepsilon_{6}^{(\alpha)} = (m_{1}s_{2} + m_{2}s_{1})^{(\alpha)}d\gamma^{(\alpha)}$$

Eq. 8.31 is shown using the correspondence above and invoking that

$$d\varepsilon_{ij}^{(\alpha)} = \frac{d\gamma^{(\alpha)}}{2} \left(m_j s_i + m_i s_j \right)^{(\alpha)}$$

14. Use Eq. 8.25 to show that no combination of slip systems can produce dilatation.

SOLUTION:

Eq.. 8.25 states that
$$d\varepsilon_{ij} = \sum_{\alpha} \frac{d\gamma^{(\alpha)}}{2} (m_j s_i + m_i s_j)^{(\alpha)}$$

Since $d\epsilon_{11} + d\epsilon_{22} + d\epsilon_{33}$ is the incremental volume change per unit volume, the above relation is used to find the volume change associated with activated slip systems,

$$d\varepsilon_{11} + d\varepsilon_{22} + d\varepsilon_{33} = \sum_{\alpha} d\gamma^{(\alpha)} (m_1 s_1 + m_2 s_2 + m_3 s_3)^{(\alpha)}$$
$$= \sum_{\alpha} d\gamma^{(\alpha)} \underline{m}^{(\alpha)} \cdot \underline{s}^{(\alpha)}$$

Since $\underline{\mathbf{m}} \cdot \underline{\mathbf{s}} = 0$ for each slip system α , the above relation indicates that $d\epsilon_{11} + d\epsilon_{22} + d\epsilon_{33} = 0$. 15. Of the twelve slip systems listed in Table 8.1, choose slip systems 1, 2, and 6. Determine whether

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these 3 slip systems are independent.

SOLUTION:

Using Eq. 8.23, the respective strain increments contributed by slip systems 1, 2, and 6 in Table 8.1 are

$$\begin{bmatrix} \varepsilon_{ij}^{(1)} \end{bmatrix} = \frac{d\gamma^{(1)}}{2\sqrt{6}} \begin{bmatrix} 2 & 0 & 1 \\ 0 & -2 & -1 \\ 1 & -1 & 0 \end{bmatrix}, \begin{bmatrix} \varepsilon_{ij}^{(2)} \end{bmatrix} = \frac{d\gamma^{(1)}}{2\sqrt{6}} \begin{bmatrix} 2 & 1 & 0 \\ 1 & 0 & -1 \\ 0 & -1 & -2 \end{bmatrix}, \begin{bmatrix} \varepsilon_{ij}^{(6)} \end{bmatrix} = \frac{d\gamma^{(1)}}{2\sqrt{6}} \begin{bmatrix} -2 & 0 & 1 \\ 0 & 2 & 1 \\ 1 & 1 & 0 \end{bmatrix}$$

or equivalently, the short-hand notation, $d\epsilon_i^{(\alpha)} = d\gamma^{(\alpha)}n_i^{(\alpha)}$, from Eq. 8.22 may be used, where the

The systems are linearly independent if no values of x and y can satisfy $x\epsilon_{ij}^{(1)} + y\epsilon_{ij}^{(2)} = \epsilon_{ij}^{(6)}$, for all components ij, or equivalently, if no values of x and y can be found that satisfy $xn_i^{(1)} + yn_i^{(2)} = n_i^{(6)}$ for all components i. After some examination, x = -1 and y = 0 appear to satisfy the above relations for all components except $d\epsilon_{13}$ (or equivalently, $d\epsilon_5$). Accordingly, these three slip systems are independent, since no combination of x and y can be found to satisfy the above relations.

16. Suppose you load in tension along the [3 8 6] direction in a f.c.c. material in which the candidate slip systems are of the type listed in Table 8.1. The crystal begins to yield at a tensile stress of 10MPa. Assume that all slip systems require the same critical resolved shear stress, τ_c , for activation. What slip system will be activated first? What is the value of τ_c ? To what crystallographic direction will the tensile axis rotate? What is the axis about which the tensile axis rotates?

SOLUTION:

The yield condition, Eq. 8.13, can be used to determine that slip system 6 in Table 8.1 will require the smallest tensile stress to activate it. The yield condition for that slip system is stated as

$$\left(\frac{[110]}{\sqrt{2}} \cdot \frac{[386]}{\sqrt{109}}\right) \left(\frac{[-111]}{\sqrt{3}} \cdot \frac{[386]}{\sqrt{109}}\right) 10 \text{MPa} = \tau_c \quad .$$

so that τ_c to yield is approximately 4.5MPa. As indicated in Eq. 8.37, the tensile axis will rotate toward the slip direction, [110]. The direction, **r**, about which the rotation occurs is given by **T** ∞ **s**, so that **r** is parallel to [-6 6 -5].

$$\dot{\beta} = -\frac{\dot{\beta}}{H_o \sin \beta_o} \sin^2 \beta$$
$$\dot{\beta} = -\dot{\epsilon}_{TT} \frac{\sin \beta}{\cos \beta}$$

17. Derive the results,

by analyzing the 2D slip plane geometry below, where $\overset{\cdot}{\mathcal{B}}$ and $\overset{\cdot}{\mathcal{E}}_{TT}$ are the increment in slip on the plane and average strain increment parallel to the current tensile axis, respectively. The other parameters are labeled in the sketch. <u>Do not</u> use the general formulation associated with the discussion of Fig. 8.5(a,b), but rather, derive the expression from a trigonometric analysis of the 2-D geometry below.



SOLUTION:

The geometry below is used, and application of the law of sines produces



Noting that $\sin(\beta_0 - \beta) = \sin\beta_0 \cos\beta - \cos\beta_0 \sin\beta$ and differentiating the second relation,

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$$\dot{\vartheta} = -\frac{H_0 \sin\beta_0}{\sin^2\beta} \dot{\beta}$$

Differentiate the first relation and then divide each side by the originating terms in the first relation. Substitute $\dot{H}/H = \dot{\xi}_{TT}$ to get

$$\dot{\beta} = -\frac{\sin\beta}{\cos\beta}\dot{\epsilon}_{TT}$$

18. Eq. 8.56 defines the increment, $d\varepsilon_{ij}^*$, in components of strain that are expressed in the reference coordinate system. Derive this relation, beginning with a statement of the Principle of Virtual Work that $\sigma_{ij}^* d\varepsilon_{ij}^* = \sum \tau^{(\beta)} d\gamma^{(\beta)}$.

SOLUTION:

We begin with a statement of the Principle of Virtual Work, $\sum \tau^{(\beta)} d\gamma^{(\beta)} = \sigma_{ij}^* d\varepsilon_{ij}^*$.

and note from Eq. 8.48 that since $\sigma_{ij}^* = \sigma_{ji}^*$, then $\tau^{(\beta)} = \frac{1}{2} \sigma_{ij}^* \left(s_i^{*(\beta)} m_j^{*(\beta)} + s_j^{*(\beta)} m_i^{*(\beta)} \right)$.

If the expression for $\tau^{(\beta)}$ is substituted in the statement of the Principle of Virtual Work, then $d\epsilon_{ij}^*$ must be defined according to Eq. 8.56.

19. Suppose you have a similar geometry as in Problem 11, but instead, there are two competing slip systems, $(\mathbf{m}^{(1)}, \mathbf{s}^{(1)})$ and $(\mathbf{m}^{(2)}, \mathbf{s}^{(2)})$. β_o is the initial angle between $\underline{s}^{(1)}$ and the initial tensile axis direction, \underline{T}_o , and γ_o is the corresponding initial angle for slip system (2).

Suppose you have documented the angle β as a function of macroscopic strain, ε_{TT} , along the T (not T_o !) axis. Explain how you could determine the relative amount of slip, $b^{(1)}$ and $b^{(2)}$, on each system. Do you have enough information?

SOLUTION:

First, we note the assumption that all $\underline{m}^{(\alpha)}$ and $\underline{s}^{(\alpha)}$ for both systems are coplanar with \underline{T} and use the geometry below.



 $\dot{\mathbf{H}} = \dot{\mathbf{b}}^{(1)} \cos \beta + \dot{\mathbf{b}}^{(2)} \cos \gamma$ First, note that

The second condition is holds since the lattice rotates rigidly, relative to the \underline{T} axis. Also note that

$$H\cos(\beta - \beta_{o}) = H_{o} + b^{(1)}\cos\beta_{o} + b^{(2)}\cos\gamma_{o}$$
$$H\sin(\beta - \beta_{o}) = b^{(1)}\sin\beta_{o} + b^{(2)}\sin\gamma_{o}$$

Each of these relations may be divided through by H_o, any occurrence of H/H_o in these relations may be replaced by exp($\bar{\epsilon}_{TT}$),. The two equations relate the two unknowns, $b^{(1)}/H_0$ and $b^{(2)}/H_0$, to the current tensile strain, $\tilde{\epsilon}_{TT}$, the starting angles γ_0 and β_0 , and the current angle β . Final expressions are left to the reader to derive.

20. Consider a material in which the critical resolved shear stress to shear a plane with normal $m^{(\alpha)}$ depends on the stress normal to the plane, according to $\tau_{c}^{(\alpha)} = \tau_{o} - \mu_{f} \left(m_{i}^{(\alpha)} \sigma_{ij} m_{j}^{(\alpha)} \right)$

where τ_0 is the critical resolved shear with zero normal load, and μ_f is a coefficient of friction. Such behavior is consistent with a hard sphere atomic picture of materials, in which the activation barrier to slide particles past one another increases with confining pressure on the material. Show that when a macroscopic stress with non-zero components σ_{11} and σ_{22} is applied, the yield condition for a slip system (α) with slip direction $s^{(\alpha)}$ and slip plane normal $m^{(\alpha)}$ is

$$\underbrace{\left(\pm s_{1}^{(\alpha)} + \mu_{f} m_{1}^{(\alpha)}\right)}_{1} m_{1}^{(\alpha)} \sigma_{11} + \underbrace{\left(\pm s_{2}^{(\alpha)} + \mu_{f} m_{2}^{(\alpha)}\right)}_{2} m_{2}^{(\alpha)} \sigma_{22} \ge \tau_{o}$$

where \pm must be interpreted as either + for both terms (+s₁(α) and +s₂(α)) or - for both terms ($s_1(\alpha)$ and $-s_2(\alpha)$).

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Further, produce a projection of the yield surface onto the σ_{11} - σ_{22} plane, assuming that σ_{11} and σ_{22} are applied along the cube directions of a f.c.c. single crystal, and that $\mu_f = 0.1$. Compare your result to the yield surface projection in Fig. 8.4.

SOLUTION:

Under such a case, the yield condition, Eq. 8.13, may be rewritten as

 $\pm s_i^{(\alpha)} \sigma_{ij} m_j^{(\alpha)} \ge \tau_o + \mu_f \left(m_i^{(\alpha)} \sigma_{ij} m_j^{(\alpha)} \right),$

and this may be rearranged into the final form,

 $\left(\pm s_1^{(\alpha)} + \mu_f m_1^{(\alpha)}\right) m_1^{(\alpha)} \sigma_{11} + \left(\pm s_2^{(\alpha)} + \mu_f m_2^{(\alpha)}\right) m_2^{(\alpha)} \sigma_{22} \ge \tau_o$

The yield condition above consists of lines in the σ_{11} - σ_{22} plane, with intercepts σ_{11}^{0} and σ_{22}^{0} along the axes given by

$$\sigma_{11}^{o(\alpha)} = \frac{\tau_{o}}{\left(\pm s_{1}^{(\alpha)} + \mu_{f}m_{1}^{(\alpha)}\right)m_{1}^{(\alpha)}}, \quad \sigma_{22}^{o(\alpha)} = \frac{\tau_{o}}{\left(\pm s_{2}^{(\alpha)} + \mu_{f}m_{2}^{(\alpha)}\right)m_{2}^{(\alpha)}}$$

The yield surface is plotted below with dashed lines, and the thicker dashed lines show the inner locus. For comparison, the result from Fig. 8.4 corresponding to $\mu_f = 0$ is shown with solid lines. The case for non-zero μ_f is distorted, so that yield in compression is different than yield in tension. This occurs since tension lowers the critical resolved shear stress for slip, while compression increases that resolved shear stress.



21. Consider the previous example where the critical resolved shear stress for a slip plane depends on

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the stress normal to the plane. Construct a yield function for this case that reduces to Eq. 8.26 when the coefficient, $\mu_f = 0$. Find the normal to the yield surface. Compare your result to $d\varepsilon_{ij}^{(\alpha)}$ predicted by Eq. 8.23. Is the strain increment normal to the yield surface?

SOLUTION:

An appropriate yield function is given by

$$\phi^{(\alpha)} = \left[\frac{1}{2}(s_i m_j + s_j m_i)^{(\alpha)} \sigma_{ij}\right]^2 - \left[\tau_0 - \mu_f m_i^{(\alpha)} \sigma_{ij} m_j^{(\alpha)}\right]^2 \quad [= 0 \text{ at yield}].$$

Taking the derivative with respect to σ_{ij} and applying that $\phi^{(\alpha)} = 0$,

$$\frac{\partial \phi^{(\alpha)}}{\partial \sigma_{ij}} = \tau_c^{(\alpha)} \left[\left(s_i m_j + s_j m_i \right)^{(\alpha)} + \mu_f m_i^{(\alpha)} m_j^{(\alpha)} \right]$$

For comparison, Eq. 8.23 states that

$$d\varepsilon_{ij} = \frac{d\gamma}{2}(m_j s_i + m_i s_j).$$

It appears that the strain increment is different than the normal to the yield surface, in that the yield surface normal contains the added component, $\mu_f m_i^{(\alpha)} m_j^{(\alpha)}$, where (α) refers to the critically stressed slip system on that portion of the yield surface.

22. Assume that you have a polycrystalline wire with a bamboo type structure, in which each grain α occupies a slice of the wire as shown below. Also, each grain has a single slip system described by slip plane normal $\mathbf{m}^{(\alpha)}$ and slip direction $\mathbf{s}^{(\alpha)}$. The critical resolved shear stress to operate any slip system is 10MPa, and the amount of slip on any plane is assumed to be negligible compared to the diameter, D, of the wire. A single angle $\beta^{(\alpha)} = \mathbf{T} \cdot \mathbf{s}^{(\alpha)}$ describes the orientation of the slip system relative to the wire axis \mathbf{T} , along which there is an applied stress, σ . Initially, the probability, P, of finding a grain with angle $\beta^{(\alpha)} = \gamma$ is the same, regardless of the value of γ , which ranges form 0 to 90°.

Sketch the probability of finding a grain with angle $\beta^{(\alpha)} = \gamma$, as a function of γ , after loading the wire to four different values of tension: $\sigma = 0$, 20, 30, and 40 MPa. To obtain your result, use a lower bound approach and make the approximation that the stress state in each grain is one of simple tension, σ . What aspect of compatibility is not satisfied here?



The assumption that the stress state in each crystal is simple tension violates compatibility across the grain boundaries in the bamboo structure. To begin, we sketch that the probability P of finding a grain with a certain angle $\beta^{(\alpha)}$ is uniform, and since β may span from 0 to 90°, the initial probability must be 1/90 for all possible angles. This is sketched below and labeled $\sigma = 0$, 20 MPa.

The analysis for other levels of stress is done by imposing that each grain that meets or exceeds the yield condition will continue to strain, and therefore rotate until the yield condition is exceeded. When the tensile stress is 20MPa, then we check for any grain orientation angle β for which the resolved shear stress exceeds 10MPa, i.e., for which

 $\tau = 20$ MPa $\cos\beta \sin\beta \ge \tau_c (= 10$ MPa)

The yield condition above is just met for any grain at angle 45°. As soon as rotation occurs, so that $\beta = 45^{\circ}-\Delta\beta$, the rotation stops, for the yield condition is no longer met. Therefore, at a tensile stress of 20MPa, grains with orientation angle $\beta = 45^{\circ}$ just meet the yield criterion. The rotation is negligible at this point, assuming that τ_c remains at 10MPa even after slip is activated.

At a tensile stress of 30MPa, the yield condition is stated as: $30 \text{ MPa} \cdot \cos \beta \cdot \sin \beta \ge 10 \text{ MPa}$

Therefore, yield is exceeded for all grains with β in between 21° and 69°. All such grains continue to rotate until $\beta = 21^{\circ}$. Thus, in a bamboo structure with 90 grains (and equally distributed with angle), 48 of those grains would displace to the 21° position, for a total probability of 49 grains out of 90 having an angle $\beta = 21^{\circ}$, zero probability for β in between 21° and 69°, and P = 1/90 for all other angles.

At a tensile stress of 40MPa, the yield condition reveals that all grains with initial angles between 15° and 75° must rotate to the 15° position. Therefore, the probability is P = 61/90 at $\beta = 15^{\circ}$, zero for all angles between 15° and 75°, and P = 1/90 for all other angles. The results are summarized in the figure below.



23. Outline the components of a computer program which will calculate the uniaxial stress-strain response of a f.c.c. single crystal loaded in tension. The tensile axis **T** is a material one, similar to the case depicted in Fig. 8.5, and initially, **T** is parallel to [1 12 11]. The sample is loaded from 0 to 40MPa over a period of 300s, so that the stress rate, $d\sigma_{TT}/dt$ is constant. Use a rate-dependent constitutive relation as described in Eqs. 8.73 and 8.74 with the following parameters:

 $\dot{\gamma}_{o}^{(\alpha)} = 10^{-3} / s$ m = 20 $\tau_{o}^{(\alpha)} = 10MPa \text{ (initially, prior to any slip)}$ Case A: $h_{\alpha\beta} = \begin{cases} 0MPa \text{ if } \alpha = \beta \\ 5MPa \text{ if } \alpha \neq \beta \end{cases}$ Case B: $h_{\alpha\beta} = \begin{cases} 5MPa \text{ if } \alpha = \beta \\ 0MPa \text{ if } \alpha \neq \beta \end{cases}$

Note that Case A simulates a larger latent hardening situation, and Case B simulates a larger self hardening situation.

Produce for each case the tensile stress-tensile strain curves along T, from $\varepsilon_{TT} = 0$ to approximately 0.8. Note the direction of T in each case when $\varepsilon_{TT} = 0.8$. For each case, discuss the dominant slip system(s) when $\varepsilon_{TT} = 0$ and when $\varepsilon_{TT} = 0.8$.

SOLUTION:

The generic outline of the computer program is to

- define the time increment, Δt , to be sufficiently small, *e.g.*, $\Delta t=0.01$ s
- define the tensile loading rate, $d\sigma_{TT}/dt = 40MPa/300s$.
- initialize $\tau_0^{(\alpha)} = 10$ MPa.
- define the unit vector T along the tensile axis to be parallel to [1 12 11]. In general, the components of all quantities will be referred to the crystal basis rather than the reference (*) basis containing T.
- define all twelve sets of $m_i^{(\alpha)}$ and $s_i^{(\alpha)}$ according to Table 8.1.

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- define the components h_{αβ} of the 12 by 12 matrix according to either of the two cases mentioned. For case A, h_{αβ} = 5MPa if α ≠ β, and h_{αβ} = 0 if α = β. For case B, h_{αβ} = 0 if α ≠ β, and h_{αβ} = 5MPa if α = β.
- begin an outer loop in which the time is incremented by Δt until the strain along the tensile axis exceeds 0.80. In this loop...

-increment σ_{TT} by $\Delta t (d\sigma_{TT}/dt)$.

-compute all $\tau^{(\alpha)}$ according to Eq. 8.13, or equivalently,

$$\tau^{(\alpha)} = \sigma_{TT} \left(\mathbf{s}^{(\alpha)} \cdot \mathbf{T} \right) \left(\mathbf{m}^{(\alpha)} \cdot \mathbf{T} \right) .$$

-calculate the increment,

$$d\gamma^{(\alpha)} = \frac{10^{-3}}{s} \left(\frac{\tau^{(\alpha)}}{\tau_o^{(\alpha)}}\right)^{20} \Delta t$$

-compute the components,

$$d\mathbf{T} = \sum d\gamma^{(\alpha)} \left(\mathbf{m}^{(\alpha)} \cdot \mathbf{T} \right) \left\{ \mathbf{s}^{(\alpha)} - \left(\mathbf{s}^{(\alpha)} \cdot \mathbf{T} \right) \mathbf{T} \right\}.$$

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-in order to define the strain increment, $d\epsilon_{TT}$, first compute the work,

 $dw = \sum |\tau^{(\alpha)} d\gamma^{(\alpha)}|$, and then define $d\varepsilon_{TT} = \frac{dw}{\sigma_{TT}}$.

-update the total strain according to $\varepsilon_{TT} = \varepsilon_{TT} + d\varepsilon_{TT}$ -update **T** according to $T_i = T_i + dT_i$

-update the $\tau_0^{(\alpha)}$ according to $\tau_0^{(\alpha)} = \tau_0^{(\alpha)} + \sum h_{\alpha\beta} d\gamma^{(\beta)}$

-print out ϵ_{TT} , σ_{TT} .

• end the outer loop

Results for Case A (larger latent hardening) and Case B (larger self hardening) are shown below. The tensile flow stress for the larger self hardening case is larger, particularly at larger strains. For both cases, slip system 6 in Table 8.1 has the larger resolved shear stress initially, and slip systems 1 and 4 have the second largest. This can be verified by noting in Fig. 8.6 that slip system 6 has the largest resolved shear stress when the tensile axis is in the vicinity of point P. For the larger latent hardening case (A), slip system 6 has the largest slip increment, $d\gamma^{(\alpha)}$ over the entire range considered, from $\varepsilon_{TT} = 0$ to 0.8. In contrast, the largest slip increment. For Case A, the components of **T** at $\varepsilon_{TT} = 0.8$ are approximately [0.51 0.80 0.31], and **T** continues to rotate toward [1 1 0]. However, the corresponding components for Case B are approximately [0.49 0.65 0.58]. The larger self hardening in Case B causes the slip system with the largest current $d\gamma^{(\alpha)}$ to harden substantially, so that no one slip system dominates over the entire history of loading.



24. Consider a polycrystal with random orientation of crystals. Produce a lower bound to yield in tension.

The simplest lower bound is to assume a trial stress distribution that is a homogeneous, simple tension state, and that does not violate yield at any point in the material. In that case, In that case, Eq. 8.42 indicates that the largest tensile stress possible is $2\tau_c$, corresponding to some optimally oriented slip system that is with $\mathbf{m} \cdot \mathbf{T} = \mathbf{s} \cdot \mathbf{T} = 1/\sqrt{2}$. Here, τ_c corresponds to the critical resolved shear stress to activate that slip system.
CHAPTER 9 - PROBLEM SOLUTIONS

A. PROFICIENCY PROBLEMS

1. For the geometry shown in Fig. 9.1b, find the relationship between the friction coefficient and the ramp angle at which sliding begins.

SOLUTION:

The equilibrium of forces is projected on the x and y axes of the figure below.



2. Use the rope formula to obtain the unknown forces in each of the following cases, assuming a friction coefficient of $\mu = 0.2$.





f:

Eq. 9.18 is used repeatedly: $F_2 = F_1 \exp(\mu\beta)$, where β is the accumulated "angle of wrap."

$$\beta = 90^\circ = \frac{\pi}{2}$$
, so that: $F_a = 100 \exp(0.2\frac{\pi}{2}) = 137 \text{ N}$
a.

$$\beta = 45^\circ = \frac{\pi}{4}$$
, so that: $F_b = 100 \exp(0.2 \frac{\pi}{4}) = 117 \text{ N}$
b:

$$\tan(\frac{\beta}{2}) = \frac{1}{2}, \quad \therefore \quad \beta = 2 \tan^{-1}(0.5) = 0.927, \text{ so that: } \mathbf{F}_{c} = 100 \exp\left[0.2(0.927)\right] = 120 \,\mathrm{N}$$

c:

$$\beta_1 (left) + \beta_2 (right) = \beta = \tan^{-1} 1 = \frac{\pi}{4}, \text{ so that } F_d = 100 \exp(0.2 \frac{\pi}{4}) = 117 \text{ N}$$
d:

e:
$$\beta_1 = \frac{\pi}{4}, \ \beta_2 = \frac{\pi}{2}, \ \beta_3 = \pi$$
, so that: $F_e = 100 \exp(0.2 \pi) = 187 \text{ N}$

$$\beta_1 = \frac{\pi}{4}, \ \beta_2 + \beta_3 = \frac{\pi}{4}, \ \therefore \ \beta = \frac{\pi}{2}, \ \text{ so that: } F_f = 100 \exp(0.2 \frac{\pi}{2}) = 137 \text{ N}$$

3. Use the geometries presented in Problem 2 and the following information to find the true strains in the region where F is applied:

 A_o = original cross sectional area of sheet = lmm^2

 $\overline{\sigma} = 200 \ \overline{\epsilon}^{0.2} \ (\text{MPa})$

Which geometry cannot be deformed as shown?

First, calculate the maximum force which can be applied to the strip without breaking it, assuming a uniaxial tensile state of stress (see Chapter 1):

$$F = \sigma A = \underbrace{200 \left(MPA = \frac{N}{mm^2} \right) \tilde{\epsilon}^{0.2}}_{\sigma} \underbrace{1mm^2 \exp(-\tilde{\epsilon})}_{A}$$

The maximum force occurs at $\varepsilon = n = 0.2$:

$$F_{\text{max}} = 200 \left(\text{MPA} = \frac{\text{N}}{\text{mm}^2} \right) 0.2^{0.2} \ 1 \text{mm}^2 \exp(-0.2) = 119 \text{ N}$$

The rope will not break only for cases b and d. For these cases, the strains may be calculated (by trial-and-error or Newton iteration, for example) as follows:

$$F_{b \text{ ord}} = 117 = 200 \exp(-\overline{\epsilon}) \overline{\epsilon}^{0.2} \implies \overline{\epsilon} = 0.13$$

4. a. A ring compression test is carried out using a standard forging press. The original and final ring dimensions are shown below. Find the value of m or μ from these data using Fig. 9.14. Height (original) = 100mm Height (final) = 60mm

Inside radius (original) = 200mm Inside radius (final) = 160mm

b. Given the purpose and use of the ring compression test, which limiting form of friction (Coulomb or sticking) is likely to be the more accurate?

SOLUTION:

a. The height and radius reductions are:

$$\frac{\Delta h}{h} = \frac{100 - 60}{100} = 40\%$$
$$\frac{\Delta r_i}{r_i} = \frac{200 - 160}{200} = 20\%$$

These values allow the use of Fig. 9.14 to determine the friction factors:

for a Coulomb friction law: $\mu \approx 0.25$

for a sticking friction or Tresca law: $m \cong 1$.

b. Direct compression will involve contact pressures near the flow strength of the material. Then, according to our simple view of friction regimes, Fig. 9.3, the friction is likely to follow more closely a sticking friction or Tresca model. To verify which friction law is more accurate, several tests should be carried out with different reduction ratios in height. With these tests, one will be able to see if the experimental measurements on the evolution of the radii follow a curve

generated by assuming either of the models presented in Fig. 9.14.
5. What are the friction factors determined for solid soap, graphite in heavy way oil, and way oil from Fig. 9.16? What difficulty do you see in applying this test to differentiate lubricant properties for production applications?

SOLUTION:

In Fig. 9.16 we observe that:

- solid soap corresponds to m = 0.1 at the beginning of the process

and m = 0.05 after;

- graphite in heavy way oil follows rather well the curve with m = 0.1;
- heavy way oil is closer in average to the m = 0.2 curve.

This test is not selective enough for a classification of the industrial lubricants, particularly at the beginning of the process.

6.a. For the double-backward extrusion test, the test is stopped at two punch strokes and the cups removed. Given the following experimental results and using Fig. 9.18, determine the friction coefficients and friction factors for this case.

$Punch\ stroke = 100mm$,	$h_1 = 80mm$,	$h_2 = 30mm$
Punch stroke $= 200mm$,	$h_1 = 120mm$,	$h_2 = 90mm$

b. What can you say about the role of sliding distance in determining friction for this configuration, *lubricant, and material combination?*

SOLUTION:

a. For a punch stroke of 100 mm we have (using Fig. 9.18):

 $\frac{h_1}{h_2} = \frac{80}{300} = 2.67$ so that $m \approx 0.15$

and for a punch stroke of 200 mm:

 $\frac{h_1}{h_2} = \frac{120}{90} = 1.33$ so that $m \cong 0.08$

b. The sliding distance is approximately constant in this test as the radii of the punches decrease so that the contact with the part takes place on a small area and not on the extruded length.

B. DEPTH PROBLEMS

7. Derive an equation or equations that could be solved numerically to find ε_1 or ε_2 as a function of

H for the test shown in the figure. Assume Coulomb friction, Hollomon hardening ($\sigma = k \epsilon^n$), and uniform tensile strain (and tensile stress) in each leg. For simplicity, assume that the pins have negligible diameter, so that the geometry can be computed simply.



Note that the wrap angle is $\beta = \tan^{-1} \frac{H}{a}$. Then, start with the rope fromula (Eq. 9.18): $F_2 = F_1 \exp(\mu\beta) = F_1 \exp\left(\mu \tan^{-1} \frac{H}{a}\right)$

$$F_{2} = \sigma_{2}A = \underbrace{k \varepsilon_{2}}_{\sigma_{2}} \underbrace{A_{0} \exp(-\varepsilon_{2})}_{\sigma_{2}}, \quad F_{1} = \sigma_{1}A = \underbrace{k \varepsilon_{1}}_{\sigma_{1}} \underbrace{A_{0} \exp(-\varepsilon_{1})}_{\sigma_{1}}$$

where:

$$k \varepsilon_2 A_0 \exp(-\varepsilon_2) = k \varepsilon_1 A_0 \exp(-\varepsilon_1) \exp(\mu\beta)$$
, or,

Then:

$$\varepsilon_2 \exp\left(-\varepsilon_2\right) = \varepsilon_1 \exp\left(-\varepsilon_1\right) \exp\left(\mu \tan^{-1}\frac{H}{a}\right)$$
 (Eq. 1)

Because this provides only one equation relating H to ε_1 and ε_2 , we must use the geometry before and after deformation to find another relationship. Assuming that the strain is uniform at all times in each leg, we find the original specimen length (2a+b) in terms of current leg lengths and strains:

$$2a + b = \sqrt{a^2 + H^2} \exp(-\varepsilon_1) + b \exp(-\varepsilon_2)$$
(Eq. 2)

Eqs. (1) and (2) may be solved simultaneously to find $\varepsilon_1(H)$ and $\varepsilon_2(H)$ for given values of m and n. Or, they may be combined to find a single equation that can also be solved numerically with the same result. First we solve for ε_2 :

$$\epsilon_{2} = -\ln\left[\frac{2a+b}{b} - \frac{\sqrt{a^{2} + H^{2}}}{b}\exp\left(-\epsilon_{1}\right), \text{ or } \right]$$

$$\epsilon_{2} = -\ln\left[\frac{2a+b}{b} - \frac{\sqrt{a^{2} + H^{2}}}{b}\exp\left(-\epsilon_{1}\right), \text{ or } \right]$$

So that finally we obtain a single equation which may be solved for $\varepsilon_1(H)$:

$$\left\{-\ln\left[\frac{2a+b}{b}-\frac{\sqrt{a^2+H^2}}{b}\exp\left(-\varepsilon_1\right)\right]\right\}^n\left[\frac{2a+b}{b}-\frac{\sqrt{a^2+H^2}}{b}\exp\left(-\varepsilon_1\right)\right] = \varepsilon_1^n e^{-\varepsilon_1} e^{\left(\mu \tan^{-1}\frac{H}{a}\right)}$$

- 8. Consider now a real test like the one shown in Problem 7.
 - a. Why would you expect to see a difference between ε_1 and ε_2 even for very narrow strips (to insure uniaxial tension) and with nearly perfect lubrication (by using rollers, Teflon, and oil, for example)? How would this effect depend or bend radius and sheet thickness?
 - b. Given results from the test described in Part a., how would you find the true friction coefficient from a similar test with fixed pins, normal lubrication, and the same material?

- a. The bending (and unbending) resistance of the sheet is responsible for an increase of the force exerted on the right and left hand sides of the sheet as it passes over the pins, and consequently $\varepsilon_1 > \varepsilon_2$. For a small radius of bending the local strain in the sheet is higher than for a large radius and so is the bending energy for a work hardening material. An increase of the sheet thickness will also increase the bending strain; therefore it will produce qualitatively the same result as a decrease of the radius of bending.
- b. The true friction contribution can be better approximated if the effect of bending can be eliminated. For that purpose it is possible in the analysis of the test to subtract the bending force measured by a frictionless test (or very low friction), with rotating pins to further reduce possible drag.
- 9. Use the rope formula and the geometry shown below to determine apparent friction coefficient from F_1 and F_2 for the modified OSU Friction Test (see Figure 9.11b and dimensions below).



SOLUTION:

We introduce notation similar to that used for Problem 7, as shown in the figure below, and make the same assumptions about small-radii pins.



The equilibrium of vertical forces and the rope formula provide two equations relating F_1 , F_2 , and F':

$$F' \sin \beta = \frac{F_1}{2}$$
, $F' = F_2 \exp(\mu\beta)$

where, as in Problem 7, $\beta = \tan^{-1} \frac{H}{a}$. F' is then eliminated from the equations, which are solved for μ to obtain the desired expression:

$$\mu = \frac{1}{\beta} \ln \left[\frac{1}{2 \sin \beta} \left(\frac{F_1}{F_2} \right) \right] = \frac{\ln \left[\frac{\sqrt{H^2 + a^2}}{2 H} \left(\frac{F_1}{F_2} \right) \right]}{\left[\tan^{-1} \left(\frac{H}{a} \right) \right]}$$

- 10. a. Explain qualitatively why and how the friction factor or friction coefficient affects the shape change in the ring compression test.
 - b. Could you design a plane-strain compression test similar to the ring compression test by simply putting a long square rod between flat platens?

SOLUTION:

- a. For a very low friction factor, the solution of the ring upsetting problem corresponds to a homogeneous strain, so that the ratio between the inner and the outer radii will remain constant. On the other hand, when the friction stress cannot be neglected, its effect will be more important on the large radius zone because of the fact that it is exerted on a larger surface and the velocity is higher. The braking effect of friction is therefore more important on the outer radius than on the inner one. That is why the inner radius increases relatively slower than the outer one, and in fact decreases for large friction stress.
- b. If the rod is sufficiently long, a zero-displacement boundary condition will be enforced by friction in the long direction. Then, we could watch the growth of the central or contact width as a function of H, or the ratio of the central width to contact width. For higher friction, the contact width will increase less relative to the central width (i.e. "barreling" will be promoted).

CHAPTER 10 - PROBLEM SOLUTIONS

A. PROFICIENCY PROBLEMS

1. Calculate the ideal work to strain a unit volume of material under uniaxial tension from $\bar{\varepsilon} = 0$ to $\bar{\varepsilon} = \varepsilon$ for each of the following hardening laws:

a. $\overline{\sigma} = K (\overline{\epsilon} + \epsilon_{o})^{n}$ (Ludwik law)b. $\overline{\sigma} = \sigma_{o} + K (\overline{\epsilon} + \epsilon_{o})^{n}$ (Swift law)c. $\overline{\sigma} = \sigma_{o} (1 - A e^{-\beta \overline{\epsilon}})$ (Voce law)d. $\overline{\sigma} = K$ (ideal plastic)e. $\overline{\sigma} = \sigma_{o} + K\overline{\epsilon}$ (linear hardening)

SOLUTION:

$$\frac{W}{V} = \int_{0}^{\varepsilon} dw = \int_{0}^{\varepsilon} \overline{\sigma} d\overline{\varepsilon} = \int_{0}^{\varepsilon} K \left(\overline{\varepsilon} + \varepsilon_{0}\right)^{n} d\overline{\varepsilon} = \frac{K}{n+1} \left(\varepsilon + \varepsilon_{0}\right)^{n+1}$$
a.

$$\frac{W}{V} = \int_{0}^{\varepsilon} \sigma_{o} d\tilde{\varepsilon} + \int_{0}^{\varepsilon} K \left(\tilde{\varepsilon} + \varepsilon_{o} \right) d\tilde{\varepsilon} = \sigma_{o} \varepsilon + \frac{K}{n+1} \left(\varepsilon + \varepsilon_{o} \right)^{n+1}$$

b.

$$\frac{W}{V} = \int_{0}^{\varepsilon} \sigma_{o} d\bar{\varepsilon} - \sigma_{o} A \int_{0}^{\varepsilon} e^{-\beta\bar{\varepsilon}} = \sigma_{o} \varepsilon + \frac{\sigma_{o} A}{\beta} \left(\varepsilon^{-\beta\varepsilon} - 1 \right)$$

c.

$$\frac{W}{V} = \int_{0}^{\varepsilon} K d\tilde{\varepsilon} = K\varepsilon$$

$$\frac{W}{V} = \int_{0}^{\varepsilon} \sigma_{o} d\tilde{\varepsilon} + K \int_{0}^{\varepsilon} K \tilde{\varepsilon} d\tilde{\varepsilon} = \sigma \varepsilon + \frac{K}{2} \varepsilon^{2}$$

2. Calculate the drawing force using the ideal work method for a wire drawing operation from 2 mm to 1 mm diameter of a Voce material where $\sigma_o = 500$ MPa, A = 0.5, $\beta = 0.2$ and where the efficiency factor is assumed to be 0.5.

SOLUTION:

e.

$$\sigma_{d} = \frac{W_{i}}{n}, \quad w_{i} = \int_{0}^{\bar{\epsilon}} \sigma_{o} \left(1 - A^{-\beta \bar{\epsilon}} \right) d\bar{\epsilon}, \quad \bar{\epsilon} = 2 \ln \left(\frac{2}{1} \right) = 1.39$$

$$\sigma_{\rm d} = \frac{1}{0.5} \left[500 (1.39) + \frac{500 (0.5)}{0.2} (\epsilon^{-0.2} (1.39) - 1) \right] = 783 \text{ MPa}$$

$$f_d = 783 \text{ MPa} \pi \left(\frac{1}{2} \text{ mm}\right)^2 = 615 \text{ N}$$
 (Note that 1 MPa = 1N / mm², or 1 MPa · mm² = 1 N

3. Calculate the extrusion force for the same conditions as Problem 2.

SOLUTION:

 $P_{ext} = \sigma_{d} = 783 \text{ MPa}, \therefore F = 783 \text{ MPa} \pi (1 \text{ mm})^2 = 2460 \text{ N}$

4. Calculate the plane-strain drawing stress to reduce sheet thickness from 2 mm to 1 mm for the material defined in Problem 2. How does this compare to the wire drawing stress computed in Problem 2?

SOLUTION:

$$\tilde{\epsilon} = \frac{2}{\sqrt{3}} \ln 2 = 0.80$$

$$\sigma_{\rm d} = \frac{1}{0.5} \left[500 (0.80) + \frac{500 (0.5)}{0.2} (\epsilon^{-0.2} (0.80) - 1) \right] = 430 \text{ MPa}$$

5. Repeat Problem 4 for plane-strain extension (instead of drawing) and compare the result with Problem 4 on an equal-original area basis.

SOLUTION:

For plane-strain extension (like uniaxial tension), there is no redundant deformation or frictional work, so $\sigma_d = w_i = 215$ MPa.

6. Calculate the total ideal work done for the operation shown below, given the hardening law shown. Assume the material obeys von Mises yield.



SOLUTION:

Volume constancy:
$$10 \ge 10 \ge 20 = 5 \ge 7.5 \ge 1$$
, $\therefore 1 = 53.3$
 $\varepsilon_1 = \ln\left(\frac{53.3}{20}\right) = 0.98$ $\varepsilon_2 = \ln\left(\frac{7.5}{10}\right) = -0.29$ $\varepsilon_3 = \ln\left(\frac{5}{10}\right) = -0.69$

$$\bar{\epsilon} = \sqrt{\frac{2}{3}} \left[(0.98)^2 + (-0.29)^2 + (-0.69)^2 + \right]^{\frac{1}{2}} = 1.0$$

$$w = V \int_{\epsilon=0}^{\epsilon=1} 500 \left[1 - 0.6 \exp(-3\epsilon) \right] =$$

= 2000mm³ $\left\{ 500 \left(1.0 \right) + \frac{500 \left(0.6 \right)}{3} \left[e^{-3(1.0)} - 1 \right] \right\}$ MPa = 810,000 N-mm, or 810 N-m

- 7. A steel deforms at high temperature at a constant effective stress of 100 MPa. For a given forming operation, the strain path may be approximated by two proportional paths, the first from $(\varepsilon_1 = 0, \varepsilon_2 = 0)$ to $(\varepsilon_1 = 0.5, \varepsilon_2 = 0.25)$ and the second path from $(\varepsilon_1 = 0.5, \varepsilon_2 = 0.25)$ to $(\varepsilon_1 = 0.6, \varepsilon_2 = 0.5)$.
 - a. What is the ideal work per volume of material?
 - b. What is the tensile strain equivalent to this forming deformation?
 - *c.* If you assumed that a single proportional path was followed from the start to finish, how would the answers to a) and b) change?

 $\sigma = 100 \text{ MPa}$

Path I: $\Delta \varepsilon_1 = 0.5$, $\Delta \varepsilon_2 = 0.25$, $\Delta \varepsilon_3 = -0.75$

$$\Delta \tilde{\epsilon}_{I} = \sqrt{\frac{2}{3}} \left[\left(0.5 \right)^{2} + \left(0.25 \right)^{2} + \left(-0.75 \right)^{2} \right]^{\frac{1}{2}} = 0.76$$

Path II: $\Delta \epsilon_1 =$

0.1,
$$\Delta \varepsilon_2 = 0.25$$
, $\Delta \varepsilon_3 = -0.35$

$$\Delta \tilde{\epsilon}_{II} = \sqrt{\frac{2}{3}} \left[(0.1)^2 + (0.25)^2 + (-0.35)^2 \right] = 0.36$$

a.
$$\frac{W_i}{V} = 100 (0.76 + 0.36) = 112 \text{ MPa}$$

b. $\epsilon_{\text{tensile}} = 1.12 (= \text{effective strain})$

$$\Delta \bar{\epsilon} = \sqrt{\frac{2}{3}} \left[(0.6)^2 + (0.5)^2 + (-1.1)^2 \right]^{\frac{1}{2}} = 1.10$$

c. for a proportional path:

$$\frac{W_i}{V} = 110 \text{ MPa} \left(\text{little different because there was no strain reversed, so nearly proportional.} \right)$$

8. *Repeat Problem 7 for a different forming operation for which the initial, intermediate, and final geometric strains are as follows:*

Initial:	$(\varepsilon_1 = 0, \varepsilon_2 = 0)$
Intermediate:	$(\varepsilon_l = 0.5, \varepsilon_2 = 0.25)$
Final:	$(\varepsilon_1 = 0, \varepsilon_2 = 0)$

$$\Delta \tilde{\epsilon}_{I} = \sqrt{\frac{2}{3}} \left[(0.5)^{2} + (0.25)^{2} + (-0.75)^{2} \right]^{\frac{1}{2}} = 0.76$$

Path I:

Path II:

a.

$$\Delta \bar{\epsilon}_{II} = \sqrt{\frac{2}{3}} \left[\left(-0.5 \right)^2 + \left(-0.25 \right)^2 + \left(0.75 \right)^2 \right]^{\frac{1}{2}} = 0.76$$

$$\frac{W_i}{V} = 100 (0.76 + 0.76) = 152 \text{ MPa}$$

b.
$$\varepsilon_{\text{tensile}} = 1.52$$

ortional path:
$$\Delta \bar{\epsilon} = \sqrt{\frac{2}{3}} \left[(0)^2 + (0)^2 + (0)^2 \right]^{\frac{1}{2}} = 0$$

c. For a proportional path:

(No ideal work done because there was no deformation, assuming a proportional path.)

a) Use L'Hôspital's Rule to find the plane-strain drawing stress for a frictionless case 9. starting from Eq. 10.29. Note that as $\mu \to 0$, $\beta \to 0$. (L'Hôspital's Rule states that the $\lim f(x)$ $\begin{array}{c}
 function \quad \left[\begin{array}{c} x \to a & \overline{g(x)} \\ x \to a & \overline{g(x)} \end{array} \right] \quad in \\
 f(x) \to \infty, \quad \lim_{x \to a} \quad g(x) \to \infty \\
 x \to o \quad x \to o \quad \end{array} \right]$ composite ofcases where limit a $\lim f(x) \to 0$, $\lim g(x) \to o$ $\int_{OP} x \rightarrow a$ x→a x→a] may be found by the ratio of the derivatives:

lim	f(x)	_ lim	f'(x)	_ f ['] (a)
x→a	$\overline{g(x)}$	$- x \rightarrow a$	$\overline{g'(x)}$	$\frac{1}{g'(a)}$

if these limits exist.

b) Compare the frictionless result obtained in part a) with the drawing stress obtained from the ideal work method.

SOLUTION:

$$\sigma_{\rm d} = \frac{H\left(1+B\right)}{B} \left[1 - \left(\frac{t_{\rm o}}{t_{\rm i}}\right)^{\rm B}\right]$$

a.

As $\mu \to 0$, $B \to 0$, so the term (1 + B) becomes 1 as $B \to 0$), i.e.

$$\begin{bmatrix} \lim_{\mu \to 0} \sigma_{d} = H \lim_{B \to 0} \left[\frac{1 - \left(\frac{t_{0}}{t_{i}} \right)^{B}}{B} \right] \end{bmatrix}$$

so: $f = H \left[1 - \left(\frac{t_{0}}{t_{i}} \right)^{B} \right]$ which goes to 0 as $B \to 0$
so: $g = B$ which goes to 0 as $B \to 0$
 $f'(0) = -H \ln \left(\frac{t_{0}}{t_{i}} \right) \left(\frac{t_{0}}{t_{i}} \right)^{B} = H \ln \left(\frac{t_{i}}{t_{0}} \right)$, and $g' = 1$ (Note: $\frac{d}{dx} a^{x} = a^{x} \ln a$)
 $\lim_{h \to 0} (t_{0})$

- $\lim_{\mu \to 0} \sigma_d = H \ln \left(\frac{t_0}{t_i} \right)$
- b. The results are identical.
- 10. Following the procedure of Exercise 10.3, derive an expression of the wire-drawing stress for the Coulomb friction case. Show all of your steps clearly.

Everything is the same except the friction law, so τ_{f} is replaced by μP in Eq. 10.3-3: $0 = r d \sigma_1 + 2 (\sigma_1 + \mu P \cot \alpha - P) dr$

The same substitution occurs in Eq. 10.3-6:

$$P = \mu P \tan \alpha - \sigma_r$$
, or $P = \frac{\sigma_r}{(\mu \tan \alpha - 1)}$

We then can remove P from the equilibrium equation:

$$0 = r d \sigma_1 + 2 (\sigma_1 + \kappa \sigma_r) dr \quad \text{where } \kappa \equiv \frac{\mu \cot \alpha - 1}{\mu \tan \alpha - 1}$$

And we use Eq. 10.3-10 as before to remove σ_{r} : $0 = r d \sigma_{1} + 2 (\sigma_{1} + \kappa \sigma_{1} - \kappa \overline{\sigma}) dr = r d \sigma_{1} + 2 [(1 + \kappa) \sigma_{1} - \kappa \overline{\sigma}] dr, \text{ or}$ $dr = -d\sigma_{r}$

$$\frac{\mathrm{d}\mathbf{r}}{\mathrm{r}} = \frac{\mathrm{d}\mathbf{\sigma}_{1}}{2\left(1+\kappa\right)\sigma_{1}-\kappa\,\overline{\sigma}} \ .$$

Integration completes the solution for a material which does not strain-harden:

$$\int_{\mathbf{r}_{i}}^{\mathbf{r}_{o}} \frac{d\mathbf{r}}{\mathbf{r}} = \int_{0}^{\sigma_{d}} \frac{d\sigma_{1}}{2(1+\kappa)\sigma_{1}-\kappa\,\overline{\sigma}}$$
$$\ln\left(\frac{\mathbf{r}_{o}}{\mathbf{r}_{i}}\right) = \frac{-1}{2(1+\kappa)} \ln\left[\frac{(1+\kappa)\sigma_{d}-\kappa\,\overline{\sigma}}{\kappa\,\overline{\sigma}}\right], \quad \text{or } \sigma = \frac{\kappa\,\overline{\sigma}}{1+\kappa} \left[1-\left(\frac{\mathbf{r}_{o}}{\mathbf{r}_{i}}\right)^{-2(1+\kappa)}\right]$$

11. Following derivations in the text and in Problems 9 and 10, complete the following table:

Calculation Type	Wire Draw	Round	Sheet Draw	Sheet Extrusion
	$\sigma_{\!d}$	Extrusion P	σ_{d}	P_{ext}
		1 ext		
Slab (general α, μ)				
Slab (any μ , small α)				
Slab ($\mu = o$)				
Ideal Work				

SOLUTION:

	Wire Draw, s _d	Round Extrusion, Pext	Sheet Draw, s _d	Sheet Extrusion, P _{ext}
Slab	$\frac{\kappa \bar{\sigma}}{1 + \kappa} \left[1 - \left(\frac{r_{o}}{r_{i}} \right)^{-2(1+\kappa)} \right]$	$\frac{\kappa \bar{\sigma}}{1 + \kappa} \left[1 - \left(\frac{r_{o}}{r_{i}} \right)^{-2(1+\kappa)} \right]$	$\frac{H\kappa}{1+\kappa} \left[1-\left(\frac{t_o}{t_i}\right)^{-(\kappa+1)}\right]$	$\frac{H \kappa}{1 + \kappa} \left[1 - \left(\frac{t_o}{t_i} \right)^{-(\kappa+1)} \right]$
Slab (small ^α)	$\frac{\left(B+1\right)}{B} \overline{\sigma} \left[1 - \left(\frac{r_{o}}{r_{i}}\right)^{2B}\right]$	$\frac{(B+1)}{B} \overline{\sigma} \left[1 - \left(\frac{r_o}{r_i} \right)^{2B} \right]$	$\frac{\mathrm{H}\left(1+\mathrm{B}\right)}{\mathrm{B}}\left[1-\left(\frac{\mathrm{t_{o}}}{\mathrm{t_{i}}}\right)^{\mathrm{B}}\right]$	$\frac{\mathrm{H}\left(1+\mathrm{B}\right)}{\mathrm{B}}\left[1-\left(\frac{\mathrm{t_{o}}}{\mathrm{t_{i}}}\right)^{\mathrm{B}}\right]$
Slab $(\mu = 0)$	$2 \overline{\sigma} \ln \left(\frac{r_o}{r_i} \right)$	$2\overline{\sigma}\ln\left(\frac{r_{o}}{r_{i}}\right)$	$HB \ln \left(\frac{t_{o}}{t_{i}}\right)$	$HB \ln \left(\frac{t_{o}}{t_{i}}\right)$
Ideal Work	$2 \overline{\sigma} \ln \left(\frac{r_o}{r_i} \right)$	$2 \overline{\sigma} \ln \left(\frac{r_o}{r_i} \right)$	HB $\ln\left(\frac{t_{o}}{t_{i}}\right)$	HB $\ln\left(\frac{t_{o}}{t_{i}}\right)$

Where: $B = \mu \cot \alpha$ $\kappa = \frac{\mu \cot \alpha + 1}{\mu \tan \alpha - 1}$ $H = \frac{2}{\sqrt{3}} \bar{\sigma}$

12. For each forming operation in Problem 11, show how much the external stress or pressure changes by using the small angle approximation.

Chapter 10

Fundamentals of Metal Forming - Solution Manual

- a. Use "typical values": r_{in} , $t_{in} = 30$ mm, r_{out} , t_{out} , t = 20mm, $\alpha = 20^{\circ}$, $\mu = -0.25$,
- b. Use "extreme values": r_{in} , $t_{in} = 40$ mm, r_{out} , t_{out} , t = 20mm, $\alpha = 45^{\circ}$, $\mu = 0.5$.

SOLUTION:

a. Typical values: $\alpha = 20^{\circ}, \ \mu = 0.25, \ \beta = \mu \cot \alpha = 0.69$ $\kappa = \frac{\mu \cot \alpha + 1}{\mu \tan \alpha - 1} = \frac{1.69}{-0.91} = -1.86$

$$\sigma_{\rm d}$$
 (wire) = $\frac{(0.69+1)}{0.69} \,\overline{\sigma} \left[1 - \left(\frac{2}{3}\right)^{2(0.69)} \right] = 1.050 \,\overline{\sigma}$

Small α :

$$\sigma_{d}$$
 (wire) = $\frac{-1.86}{1-1.86} \,\overline{\sigma} \left[1 - \left(\frac{2}{3}\right)^{-2 \left(1-1.86\right)} \right] = 1.086 \,\overline{\sigma}$

Large α :

Wire draw difference =
$$3.3\%$$

$$\sigma_{d} \text{ (sheet)} = \frac{(0.69+1)}{0.69} \text{ H} \left[1 - \left(\frac{2}{3}\right)^{(0.69)} \right] = 0.598 \text{ H}$$

Small α :

$$\sigma_{d}$$
 (sheet) = $\frac{-1.86}{1-1.86}$ H $\left[1 - \left(\frac{2}{3}\right)^{-(1-1.86)}\right]$ = 0.637 H

Large α :

Sheet draw difference =
$$6.1\%$$

b. Extreme values: $\alpha = 45^{\circ}$, $\mu = 0.5$, $\beta = \mu \cot \alpha = 0.5$ $\kappa = \frac{\mu \cot \alpha + 1}{\mu \tan \alpha - 1} = \frac{1.5}{-0.5} = -3$

$$\sigma_{\rm d} \text{ (wire)} = \frac{\left(0.5+1\right)}{0.5} \,\overline{\sigma} \left[1 - \left(\frac{1}{2}\right)^{2\left(0.5\right)}\right] = 1.50 \,\overline{\sigma}$$

Small $\alpha_{:}$

$$\sigma_{d}$$
 (wire) = $\frac{-3}{1-3} \bar{\sigma} \left[1 - \left(\frac{1}{2}\right)^{-2(1-3)} \right] = 1.41 \bar{\sigma}$

Large α :

Wire draw difference =
$$6.3\%$$

$$\sigma_{d} \text{ (sheet)} = \frac{(0.5+1)}{0.69} \text{ H} \left[1 - \left(\frac{1}{2}\right)^{(0.5)} \right] = 0.879 \text{ H}$$

Small $\alpha_{:}$

$$\sigma_{d}$$
 (sheet) = $\frac{-3}{1-3} H \left[1 - \left(\frac{1}{2}\right)^{-1} \right] = 1.125 H$

Large α :

Sheet draw difference = 22%

13. Perform a slab calculation for plane-strain compression of a Hollomon material $(\overline{\sigma} = 600 \text{ MPa} \overline{\epsilon}^{0.25})$ subject to Coulomb friction ($\mu = 0.3$). Assuming an initial geometry of $h_o = 100$ mm and $b_o = 50$ mm, plot the friction hill at several values of h/h_o (0.75, 0.5, 0.25) and find P_{max} and $P_{average}$ as a function of h/h_o .

SOLUTION:

$$P = \frac{2\overline{\sigma}}{\sqrt{3}} \exp\left[\frac{2\mu}{h} \left(\frac{b}{2} - x\right)\right], \quad \mu = 0.3, \quad h_o = 100 \text{ mm}, \quad b_o = 50 \text{ mm}$$

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$\frac{h}{h_o}$	0.75	0.5	0.25
h	75	50	25
$b\left(=\frac{h_{o}b_{o}}{h}\right)$	67	100	200
$\mathcal{E}_{h}\left(=\ln\frac{h}{h_{o}}\right)$	-0.29	-0.69	-1.39
$\bar{\mathbf{\epsilon}}\left(=\frac{2}{\sqrt{3}}\left \mathbf{\epsilon}_{\mathrm{h}}\right \right)$	0.33	0.80	1.61
$\bar{\sigma} \left(= 600 \ \epsilon^{-0.25}\right)$	455 MPa	567 MPa	676 MPa

h, b, σ are found as follows:





14. Consider the forging operation shown below. If the die material can withstand a contact pressure of 1500 MPa and the workpiece exhibits a constant effective stress of 500 MPa, what is the minimum height to which the billet can be forged. Assume two cases: a) $\mu = 0.25$, b) sticking friction.



a. Coulomb friction, $\mu = 0.25$: $P_{max} = 1500 \text{ MPa} = \frac{2 \overline{\sigma}}{\sqrt{3}} \exp \frac{\mu b}{h}$

Constant volume:
$$h_o b_o = h b$$
, $b = \frac{h_o b_o}{h} = \frac{900 \text{mm}^2}{h}$

1500 MPa =
$$\frac{2(500 \text{ MPa})}{\sqrt{3}} \exp\left(\frac{0.3 \text{ b}_{0} \text{ h}_{0}}{\text{ h}^{2}}\right)$$

$$\therefore h_{\min} = \left[\frac{900 \text{mm}^2}{\sqrt{3} \frac{1500 \text{ MPa}}{500 \text{ MPa}} - 2}\right]^{\frac{1}{2}} = 16.8 \text{mm}$$

b. Sticking friction,
$$P_{max} = 1500 \text{ MPa} = \frac{2\overline{\sigma}}{\sqrt{3}} \left(1 + \frac{b}{2h}\right)$$

$$\therefore h_{\min} = \left[\frac{900 \text{mm}^2}{\sqrt{3} \frac{1500 \text{ MPa}}{500 \text{ MPa}} - 2}\right]^{\frac{1}{2}} = 16.8 \text{mm}$$

 15. Consider the drawing of a sheet which reduces both of its dimensions in the cross sections. Set up and derive the differential equation which governs the drawing operation. Show all of your work and leave the resulting equation in the form of the following variables:
 P2, P3, dA2, dA3, α, β, μ, t, and σ1

Considering the symmetry of the operation and assuming a von Mises material, find the ratio of the stresses $\sigma_1/\sigma_2/\sigma_3$.



SOLUTION:

Force balance in the longitudinal direction:

$$0 = d \left(\sigma_1 2 t^2\right) + 2 P_2 \left(\mu \cos \alpha + \sin \alpha\right) d A_2 + 2 P_3 \left(\mu \cos \beta + \sin \beta\right) d A_3$$

The strain ratios may be found from the geometry:

$$\varepsilon_{\rm h} = \ln \frac{t}{30}$$
 $\varepsilon_{\rm w} = \ln \frac{2t}{60} = \ln \frac{t}{30}$ $\varepsilon_{\rm l} = -2\ln \frac{t}{30}$, so

 $\epsilon_{\rm h}/\epsilon_{\rm w}/\epsilon_{\rm l}=-1/-1/2$, (same as uniaxial tension)

Therefore, the stress state must be the one for uniaxial tension, or one which differs from that by a hydrostatic pressure (which does not affect plastic deformation of a von Mises material):

 $\therefore \sigma_1 / \sigma_2 / \sigma_3 = \bar{\sigma} \pm P / \pm P / \pm P$

B. DEPTH PROBLEMS

16. Find the efficiency of a wire drawing operation, as estimated by a slab calculation, assuming the following parameters:

 $\overline{\sigma} = 1000 \text{ MPa}, \ \alpha = 10^{\circ}, \ r_{in} = 50 \text{ mm}, \ r_{out} = 40 \text{ mm}, \ \mu = 0.25.$

SOLUTION:

$$\sigma_{d} = \frac{B+1}{B} \,\overline{\sigma} \left[1 - \left(\frac{r_{o}}{r_{i}} \right)^{2B} \right]$$

where $B = \mu \cot \alpha$ (small α) = 0.25 cot 10° = 1.42

$$\sigma_{\rm d} = \frac{242}{1.42} \ 1000 \ \text{MPa} \left[1 - \left(\frac{4}{5}\right)^{2.84} \right] = 800 \ \text{MPa}$$

$$\sigma_{\rm d}$$
 (ideal) = $\frac{W_{\rm i}}{\rm Vol} = \int_{0}^{\bar{\epsilon}} \bar{\sigma} \, d\bar{\epsilon} = 1000 \, \rm MPa \, 2 \, \ln\left(\frac{5}{4}\right) = 446 \, \rm MPa$

Ideal work calculation:

Wire draw slab solution:

Efficiency =
$$\frac{W_i}{W_a} = \frac{\sigma_d (ideal)}{\sigma_d (slab)} = \frac{446 \text{ MPa}}{800 \text{ MPa}} = 56\%$$

17. For the conditions presented in Problem 16, what is the maximum r_{out} that can be performed?

SOLUTION:

Maximum draw stress = 1000 MPa

1000 MPa =
$$\frac{2.42}{1.42}$$
 1000 MPa $\left[1 - \left(\frac{r_o}{r_i}\right)^{2.84}\right]$

$$\left(\frac{r_{o}}{r_{i}}\right)_{min} = \left(1 - \frac{1.42}{2.42}\right)^{\frac{1}{2.84}} = 0.73, \text{ or } \left(\frac{r_{o}}{r_{i}}\right)_{max} = 1.37$$

18. a. Repeat Exercise 10.2 for the most general assumption that the redundant work is a fixed fraction of the friction work:

$$\alpha = \frac{w_r}{w_f}, \quad 0 \le \alpha \le 1.$$

- b. Show that your general expression reduces to the standard expression (Eq. 10.13) when $\alpha = 0$.
- c. Show that your expression reduces to the improved expression (Eq. 10.2-10) when $\alpha = 1$. $(d_i)^*$
- d. Compute $\overline{(d_0)}$ for the cases in Table 10.2 using your general expression with $\alpha \to \infty$ (i.e. all non-ideal work is redundant work.) What are the differences compared with standard expression for these cases?

e. Use your result and your knowledge of scaling of w_f and w_r with respect to size to predict $\frac{d_i}{d_i}^*$

whether $\left(\frac{d_i}{d_o}\right)^*$ would go up or down as the size of the drawing operation increased. (Hint: how does α depend on size?)

SOLUTION:

a.

b.

$$\sigma_{d} = \frac{W_{a}^{\text{total}}}{\text{Vol}} = \frac{W_{i}}{\eta} = \frac{1}{\eta} \frac{k}{n+1} \bar{\epsilon}^{n+1}$$

$$\overline{\sigma}_{a} = k \overline{\epsilon}_{a}^{n}$$
, where $\overline{\epsilon}_{a}$ is such that $W_{i} + W_{r} = W_{def} = \frac{\kappa}{n+1} \overline{\epsilon}_{a}^{n+1}$

$$\begin{split} \mathbf{W}_{a} &= \frac{\mathbf{W}_{i}}{\eta} = \mathbf{W}_{i} + \mathbf{W}_{f} + \mathbf{W}_{r} = \mathbf{W}_{i} + \left(\frac{1}{\alpha} + 1\right) \mathbf{W}_{r} \\ \text{, or} \\ \mathbf{W}_{r} &= \left(\frac{1-\eta}{\eta}\right) \left(\frac{\alpha}{1+\alpha}\right) \mathbf{W}_{i}, \ \mathbf{W}_{def} = \left[1 + \left(\frac{1-\eta}{\eta}\right) \left(\frac{\alpha}{1+\alpha}\right)\right] \mathbf{W}_{i} \equiv \beta \mathbf{W}_{i} \\ \mathbf{\tilde{\epsilon}}_{a} &= \left[\left(\frac{n+1}{k}\right) \beta \mathbf{W}_{i}\right]^{\frac{1}{n+1}}, \ = \text{ where } \mathbf{W}_{i} = \left(\frac{k}{n+1}\right) \mathbf{\tilde{\epsilon}}_{i}^{n+1} \end{split}$$

$$\boldsymbol{\bar{\sigma}}_{a} \ = \ \kappa \left[\left(\frac{n+1}{k} \right) \boldsymbol{\beta} \boldsymbol{W}_{i} \right]^{\frac{n}{n+1}} \ = \ k \ \boldsymbol{\beta}^{\frac{n}{n+1}} \ \boldsymbol{\bar{\epsilon}}_{i}^{\ n}$$

LDR equation: $\bar{\sigma}_{a} = \sigma_{d}$, $k \beta^{n'(n+1)} \bar{\epsilon}^{n} = \frac{1}{\eta} \frac{k}{n+1} \bar{\epsilon}^{n+1}$, $\bar{\epsilon}_{i} = \eta (n+1) \beta^{n'(n+1)}$

$$\left(\frac{r_{o}}{r_{i}}\right)^{*} = \exp\left[\frac{\eta}{2}\left(n+1\right)\beta^{\frac{n}{n+1}}\right] = \exp\left\{\frac{\eta}{2}\left(n+1\right)\left[1+\left(\frac{1-\eta}{\eta}\right)\left(\frac{\alpha}{1+\alpha}\right)\right]^{\frac{n}{n+1}}\right\}$$

for $\alpha = 0, \ \beta = 1, \ \text{LDR} = \exp\left[\frac{\eta}{2}\left(n+1\right)\right]$ (same as Eq. 10.13b)

for
$$\alpha = 1$$
, $\beta = \left(\frac{\eta + 1}{2\eta}\right)$, $LDR = \exp\left[\frac{\eta}{2}\left(n + 1\right)\left(\frac{\eta + 1}{2\eta}\right)^{\frac{n}{n+1}}\right]$ (same as Eq. 10.2-10)

for $\alpha = \infty$, $\beta = \frac{1}{\eta}$, LDR = exp $\left[\frac{\eta + 1}{2}\right]$ (same as Eq. 10.13b with $\eta = 1$ or Eq. 10.2-10)

Efficiency, material	η	n	$ \begin{pmatrix} \frac{d_i}{d_o} \end{pmatrix}_{std.}^* $ (Eq. 10.13)	$\left(\frac{d_i}{d_o}\right)^*_{improved}$ $\alpha = \infty$	Difference (%)
Ideal, brass	1.00	0.50	2.12	2.12	0%
Ideal, steel	1.00	0.25	1.87	1.87	0%
Medium, steel	0.70	0.25	1.55	1.87	21%
Low, steel	0.50	0.25	1.37	1.87	36%
Low, aluminum	0.50	0.15	1.33	1.78	34%
V. Low, zinc	0.25	0.00	1.15	1.65	43%

e. Imagine an operation with $r_i = 1$ and draw length l = 1, and imagine scaling this operation by a factor C, i.e. $r_i = C$, l = C. The deformation work will be proportional to the volume deformed, i.e. $W_{def} = (scale) = C^3 W_{def}$. The friction work will depend on the swept surface, i.e. $W_f = (scale) = C^2 W_f$. Therefore,

for
$$\alpha = \frac{W_r}{W_f} \to \infty$$
 as $C \to \infty$ (large operations) and $\alpha \to 0$ as $C \to$ (small operations).

Therefore, larger drawing operations can achieve somewhat larger draw ratios because friction plays a smaller role.

- 19. a) Derive equations similar to Eqs. 10.57 for the case of sticking friction.
 - b) Assuming plane-strain conditions, eliminate P from your equations.
 - c) By making appropriate geometric substitutions, write your equations in terms of t and dt alone, and α and $d\alpha$ alone.

SOLUTION:

a. For sticking friction, the friction stress is $\tau_f = \frac{\overline{\sigma}}{\sqrt{3}}$ whereas with Coulomb friction $\tau_f = \mu P$. With this substitution, Eqs. 10.57a and b may be immediately rewritten:

$$0 = d \left(\sigma_1 t\right) + \left(\frac{2\overline{\sigma}}{\sqrt{3}} - 2P \tan \alpha\right) d x_1 \quad \text{before N (Neutral Point)}$$

$$0 = d \left(\sigma_1 t \right) + \left(-\frac{2\overline{\sigma}}{\sqrt{3}} - 2P \tan \alpha \right) dx_1 \quad \text{after N (Neutral Point)}$$

b. For plane-strain conditions (see Eq. 10.32), and small α :

$$P = \sigma_1 + \frac{2}{\sqrt{3}} \bar{\sigma}, \text{ so } 0 = d(\sigma_1 t) + 2 \left[\pm \frac{\bar{\sigma}}{\sqrt{3}} \sigma_1 - \sigma_1 \tan \alpha - \frac{2\bar{\sigma}}{\sqrt{3}} \tan \alpha \right] dx_1$$

where the plus sign is taken before the neutral point and the minus is taken after it.

c. Using the geometry shown, and defining a new variable $t^{'}$ as shown, the following geometric relationships may be obtained readily:

$$t' \equiv \frac{t - t_o}{2} = R - R \cos \alpha \qquad dt' = \frac{dt}{2} = R \sin \alpha \, d\alpha$$
$$x_1 = -\sqrt{2R t' - t'^2} = -R \sin \alpha \qquad dx_1 = -\frac{(R - t) dt'}{\sqrt{2R t' - t'^2}} = -R \cos \alpha \, d\alpha$$

$$t = 2(R - R\cos\alpha) + t_{o}$$
 $dt = 2R\sin\alpha d\alpha$ $\tan\alpha = \frac{\sqrt{2Rt'-t'^{2}}}{R-t'}$

Then, in order to write the governing equation in terms of α and $d\alpha$ alone, we make the appropriate substitutions in the result of Part c:

$$0 = \underbrace{d(\sigma_{1} t)}_{0} + 2 \left[\pm \frac{\overline{\sigma}}{\sqrt{3}} - \sigma_{1} \tan \alpha - \frac{2 \overline{\sigma}}{\sqrt{3}} \tan \alpha \right] \underbrace{dx_{1}}_{-R \cos \alpha} d\alpha$$
$$d \left\{ \sigma_{1} \left[2(R - R \cos \alpha) + t_{o} \right] \right\} - R \cos \alpha d\alpha$$

which, when rearranged, obtains:

$$0 = \sigma_1 R \sin \alpha \, d\alpha + \left[2 \left(R - R \cos \alpha \right) + t_o \right] d\sigma_1 - 2R \left[\pm \frac{\overline{\sigma}}{\sqrt{3}} \cos \alpha - \sigma_1 \sin \alpha - \frac{2 \overline{\sigma}}{\sqrt{3}} \sin \alpha \right] d\alpha$$



In order the write the same expression in terms of t and dt alone, we make a different set of substitutions:

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$$0 = \underbrace{d(\sigma_{t}t)}_{\tan \alpha} + 2\left[\pm \frac{\overline{\sigma}}{\sqrt{3}} - \sigma_{1} \tan \alpha - \frac{2 \overline{\sigma}}{\sqrt{3}} \tan \alpha\right] \underbrace{dx_{1}}_{\tan \alpha}$$
$$\tan \alpha = \frac{\sqrt{2 R t' - t'^{2}}}{R - t'} \qquad dx_{1} = -\frac{(R - t) dt'}{\sqrt{2 R t' - t'^{2}}}$$

which, when rearranged, obtains:

$$0 = d(\sigma_1 t) + 2\left[\pm \frac{\overline{\sigma}}{\sqrt{3}} \frac{(R-t)}{\sqrt{2Rt'-t'^2}} + \sigma_1 + \frac{2\overline{\sigma}}{\sqrt{3}} \right] dt'$$

where the minus sign applies before the neutral point and the plus sign afterward, and $t' = \frac{t - to}{2}$ and dt' = dt/2.

20. Real rolling operations exhibit several characteristics which are not easily seen in simple slab analysis. What do you think is the origin of the effects?



SOLUTION:

- a. Friction at the rolls draws the sheet through the rolling operation, but since it operates only at the surface, the central material tends to be restrained by the material behind it.
- b. The shear strains established by the gradient of friction force can cause shear banding, and strain localization through the thickness. In fact, the tensile stress components along the rolling direction can also be large enough to produce cracking.
- c. Widening occurs when the roll contact zone is comparable in the rolling direction relative to the

width of the contract region, so that the operation begins to resemble an upsetting operation.

- d. Side cracking can occur because shear strains are set up as the plane strain state in the center approaches uniaxial compression at the edge. The tensile elongation caused by adjacent material can also introduce tensile stress cracking.
- e. Crowning occurs by roll bending, i.e. the pressure across the roll causes beam-type bending of the rolls.
- 21. Consider a sheet-drawing operation (plane strain) that uses streamlined dies, as shown.



a) Use a simple 5-step numerical procedure to find σ_1 at t = 40mm, 36mm, 32mm, 28 mm, and 24 mm and 20mm.

b) What is the limiting t_{out} ratio that can be attained with these dies based on your numerical procedure? How does this compare with the result for straight-sided dies?

SOLUTION:

 $d \sigma_1$

a.

$$= \left[\mathbf{B} \, \boldsymbol{\sigma}_1 - \mathbf{H} \left(\mathbf{1} + \mathbf{B} \right) \right] \frac{\mathrm{d}t}{\mathrm{t}} \quad \text{(Eq. 10.25)}$$

$$\Delta \sigma_1 = \left[B \sigma_1 - H \left(1 + B \right) \right] \frac{\Delta t}{t} \text{ where } H = 100 \text{ ksi and } B = 0.25 \text{ cot } [2 (t-18)]$$

Step #	t(mm)	В	Δt (mm)	$\Delta \sigma_1$ (KSi)	σ_1 (ksi)
0	40				0 (bdy. cond.)
1	36	0.34	-4	14.9	14.9
2	32	0.47	-4	17.5	32.4
3	28	0.69	-4	20.1	53.4
4	24	1.18	-4	25.9	79.3
5	20	3.58	-4	34.9	114.2

so, $\sigma_d = 114.2$ ksi.

b. Interpolating between steps 4 and 5 to find the t where $\sigma_d = 100$ KS: yields

$$t^* = 24 - \frac{100 - 79.3}{114.2 - 79.3} \cdot 4mm = 21.6mm, \text{ or } \left(\frac{t_i}{t_o}\right)^* = \left(\frac{40}{21.6}\right) = 1.85$$

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22. Using a slab analysis similar to the one for the plane-strain compression case, derive an expression for pressure as a function of radial position for the compression of a cylinder. Consider cases with a) Coulomb friction and b) sticking friction.

SOLUTION:



Force balance in the radial direction:

 $\left(\sigma_{r} + d \sigma_{r}\right) h \left(r + dr\right) d\theta - \sigma_{r} h r d \theta - 2 \sigma_{\theta} h d r \frac{d \theta}{2} - 2 \tau_{f} r d r d \theta = 0$ $r h d \sigma_{r} + \left(\sigma_{r} h - \sigma_{\theta} h - 2 2 \tau_{f}\right) dr = 0$

For axisymmetric flow: $\varepsilon_r = \varepsilon_{\theta}$, so $\sigma_r = \sigma_{\theta}$

For isotropic plasticity: $h d \sigma_r = 2 \tau_f d r$

For von Mises flow rule: $\sigma_r = \sigma_z + \bar{\sigma} = \bar{\sigma} - P$

a. Coulomb Case: $\tau_{f} = \mu P$

$$-h d P = 2 \mu P d r \implies \frac{dP}{P} = \frac{2\mu}{h} d r$$
$$-\int_{P}^{\overline{\sigma}} \frac{dP}{P} = \int_{r}^{R} \frac{2\mu}{h} d r \implies \ln \frac{P}{\overline{\sigma}} = \frac{2\mu}{h} (R-r), \text{ so}$$
$$P = \overline{\sigma} \exp\left[\frac{2\mu}{h} (R-r)\right] \qquad \text{(Coulomb friction)}$$

b. Sticking Friction Case: $\tau_{\rm f} = \frac{\bar{\sigma}}{\sqrt{3}}$

$$-h dP = \frac{2}{\sqrt{3}} \bar{\sigma} dr \implies dP = -\frac{2}{\sqrt{3}h} \bar{\sigma} dr \int_{P}^{\bar{\sigma}} dP = \int_{r}^{R} \frac{2}{\sqrt{3}} h \bar{\sigma} dr$$
$$P = \bar{\sigma} + \frac{2\bar{\sigma}}{\sqrt{3}h} (R - r) \text{ (Sticking friction)}$$

23. How would the sheet drawing stresses depend on normal plastic anisotropy (r) based on Hill's theory?

SOLUTION:

To find the effect of normal anisotropy, we need to substitute Hill's normal quadratic yield function (Eq. 7.60) in place of von Mises yield, and re-solve for σ_d . [Note, however that we conventionally choose σ_3 normal to the sheet plane instead of x_2 as was done previously for the sheet-draw problem.]

For plane-strain ($\varepsilon_2 = 0$) using Eq. 7.61b:

$$d \varepsilon_2 = 0 = \left[\left(\sigma_2 - \sigma_3 \right) - r \left(\sigma_1 - \sigma_2 \right) \right] \frac{d \varepsilon}{\left(1 + r \right) \sigma}, \text{ so } \qquad \sigma_2 = \frac{r}{1 + r} \sigma_1 + \frac{1}{1 + r} \sigma_3$$

Substituting into Eq. 7.60 (Hill's yield function):

$$\overline{\sigma} = \frac{\sqrt{2r+1}}{1+r} \left| \sigma_1 - \sigma_3 \right|$$
(Hill)

Which corresponds to the von Mises result (Eq. 10.22) for r = 1.

$$\overline{\sigma} = \frac{\sqrt{3}}{2} \left| \sigma_1 - \sigma_3 \right|$$
 (von Mises)

Therefore, the replacement for Eq. 10.23 becomes

$$\sigma_3 = \sigma_1 - H_{\text{Hill}}$$
, where $H_{\text{Hill}} = \frac{1+r}{\sqrt{2r+1}} \bar{\sigma}$

The final result corresponds to Eq. 10.29:

$$\sigma_{d} = \frac{H_{Hill} (1 + B)}{B} \left[1 + \left(\frac{t_{o}}{t_{i}} \right)^{B} \right], \quad H_{Hill} = \frac{1 + r}{\sqrt{2r + 1}} \,\overline{\sigma}$$

Therefore as r increases, σ_d increases.